Chapter 8
Dimensional Analysis and Similitude

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Significant learning outcomes

Conceptual Knowledge

- State the Buckingham Π theorem.
- Identify and explain the significance of the common π-groups.
- Distinguish between model and prototype.
- Explain the concepts of dynamic and geometric similitude.

Procedural Knowledge

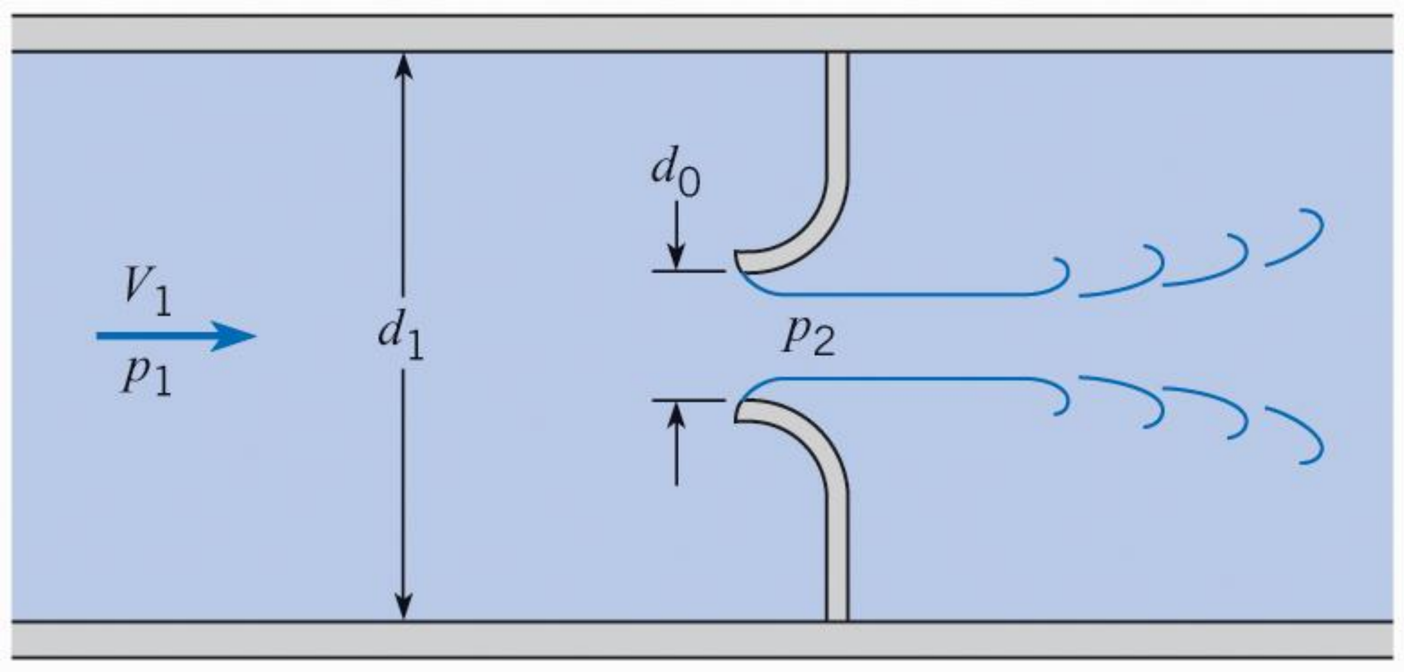
- Apply the Buckingham Π theorem to determine number of dimensionless variables.
- Apply the step-by-step procedure to determine the dimensionless π-groups.
- Apply the exponent method to determine the dimensionless π-groups.
- Distinguish the significant π-groups for a given a flow problem.

Applications (typical)

- Drag force on a blimp from model testing.
- Ship model tests to evaluate wave and friction drag.
- Pressure drop in a prototype nozzle from model measurements.

8.1 Need for dimensional analysis

- Experimental studies in fluid problems
- Model and prototype
- Example: Flow through inverted nozzle

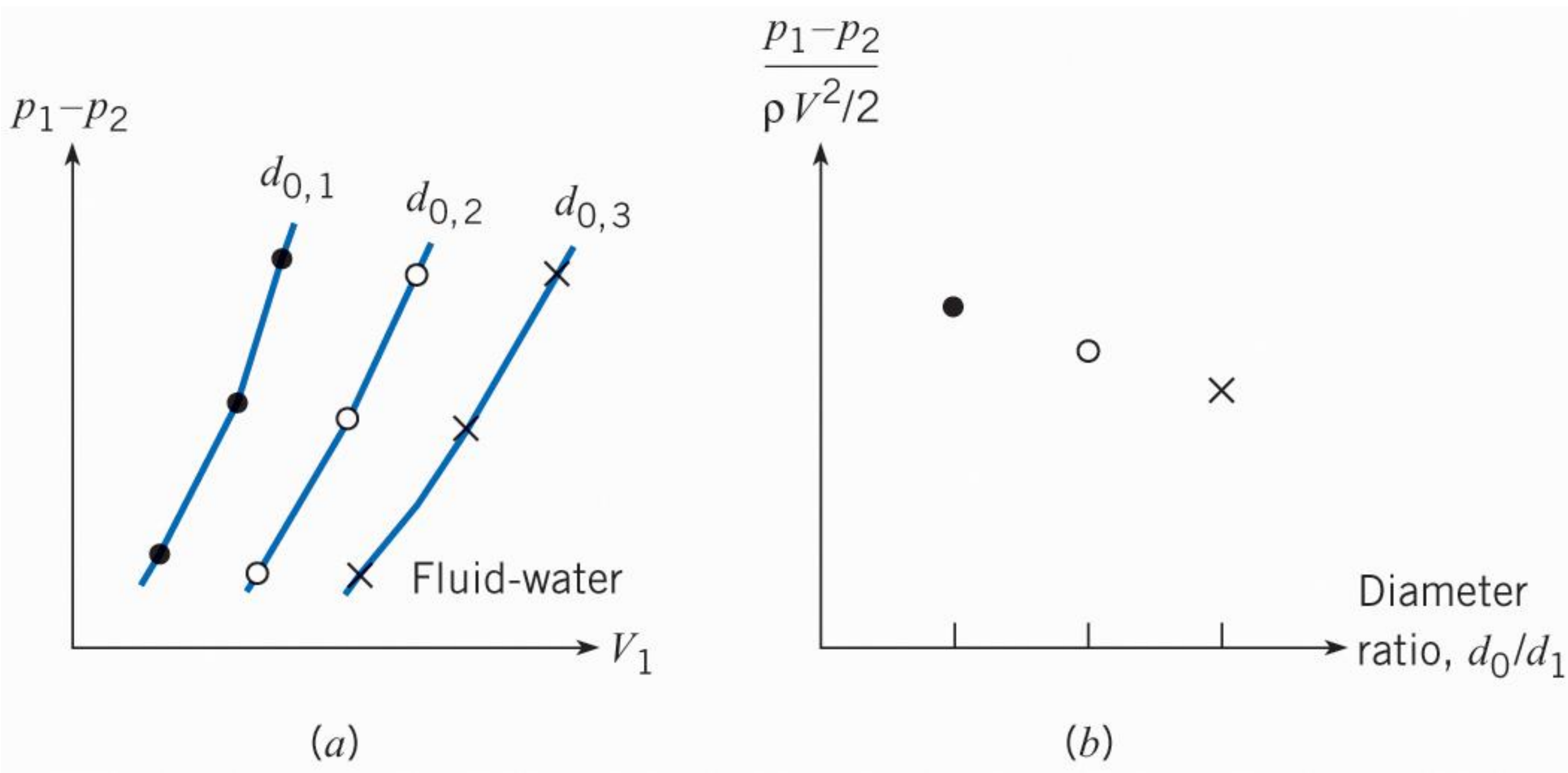


Pressure drop through the nozzle can shown as:

$$\frac{p_1 - p_2}{(\rho V^2)/2} = f\left(\frac{d_0}{d_1}, \frac{\rho V_1 d_0}{\mu}\right)$$

For higher Reynolds numbers

$$\frac{p_1 - p_2}{(\rho V^2)/2} = f\left(\frac{d_0}{d_1}\right)$$



8.2 Buckingham pi theorem

In 1915 Buckingham showed that the number of independent dimensionless groups of variables (dimensionless parameters) needed to correlate the variables in a given process is equal to *n - m,* where *n* is the number of variables involved and *m* is the number of basic dimensions included in the variables.
Buckingham referred to the dimensionless groups as Π, which is the reason the theorem is called the Π theorem.

$$y_1 = f(y_2, y_3, y_4, \ldots\ldots y_n)$$

$$\pi_1 = \varphi(\pi_2, \pi_3, \ldots\ldots \pi_{n-m})$$

For example, the drag force F of a fluid past a sphere would be a function of the velocity V, mass density ρ, viscosity μ, and diameter D, a total of five variables (F, V, ρ, μ, D) and three basic dimensions (L, M and T) are involved.
By Buckingham pi theorem there will be 5-3=2 π groups.
So, these two groups can be correlated as:

$$\pi_1 = \varphi(\pi_2)$$

8.3 Dimensional analysis

Purposes of dimensional analysis

- ►To generate nondimensional parameters that help in the design of experiments (physical and/or numerical) and in the reporting of experimental results
- ►To obtain scaling laws so that prototype performance can be predicted from model performance
- ►To (sometimes) predict trends in the relationship between parameters

8.3 Dimensional analysis

Step by step method

1. Identify the significant dimensional variables and write out the primary dimensions of each.
2. Apply the Buckingham Π theorem to find the number of π-groups.
3. Set up table with the number of rows equal to the number of dimensional variables and the number of columns equal to the number of basic dimensions plus one ($m + 1$).
4. List all the dimensional variables in the first column with primary dimensions.

8.3 Dimensional analysis

Step by step method

5. Select a dimension to be eliminated, choose a variable with that dimension in the first column, and combine with remaining variables to eliminate the dimension. List combined variables in the second column with remaining primary dimensions.
6. Select another dimension to be eliminated, choose from variables in the second column that has that dimension, and combine with the remaining variables. List the new combinations with remaining primary dimensions in the third column
7. Repeat Step 6 until all dimensions are eliminated. The remaining dimensionless groups are the π-groups. List the π-groups in the last column

EXAMPLE 8.1 Π-GROUPS FOR BODY FALLING IN A VACUUM

There are three significant dimensional variables for a body falling in a vacuum (no viscous effects): the velocity V; the acceleration due to gravity, g; and the distance through which the body falls, h. Find the π-groups using the step-by-step method.

Problem Definition

Situation: Body falling in vacuum, $V = f(g, h)$.

Find: π-groups.

Plan

Follow procedure for step-by-step method in Table 8.1.

Variable	[ ]	Variable	[ ]	Variable	[ ]
V	$\frac{L}{T}$	$\frac{V}{h}$	$\frac{1}{T}$	$\frac{V}{\sqrt{gh}}$	0
g	$\frac{L}{T^2}$	$\frac{g}{h}$	$\frac{1}{T^2}$		
h	L				

EXAMPLE 8.2 π-GROUPS FOR DRAG ON A SPHERE USING STEP-BY-STEP METHOD

The drag F_D of a sphere in a fluid flowing past the sphere is a function of the viscosity μ, the mass density ρ, the velocity of flow V, and the diameter of the sphere D. Use the step-by-step method to find the π-groups.

Problem Definition

Situation: Given $F_D = f(V, \rho, \mu, D)$.

Find: The π-groups using the step-by-step method.

Variable	[ ]	Variable	[ ]	Variable	[ ]	Variable	[ ]
F_D	$\frac{ML}{T^2}$	$\frac{F_D}{D}$	$\frac{M}{T^2}$	$\frac{F_D}{\rho D^4}$	$\frac{1}{T^2}$	$\frac{F_D}{\rho V^2 D^2}$	0
V	$\frac{L}{T}$	$\frac{V}{D}$	$\frac{1}{T}$	$\frac{V}{D}$	$\frac{1}{T}$		
ρ	$\frac{M}{L^3}$	ρD^3	M				
μ	$\frac{M}{LT}$	μD	$\frac{M}{T}$	$\frac{\mu}{\rho D^2}$	$\frac{1}{T}$	$\frac{\mu}{\rho VD}$	0
D	L						

Table 8.2 THE EXPONENT METHOD

Step	Action Taken During This Step
1	Identify the significant dimensional variables, y_i, and write out the primary dimensions of each, $[y_i]$.
2	Apply the Buckingham Π theorem to find the number of π-groups.
3	Write out the product of the primary dimensions in the form $[y_1] = [y_2]^a \times [y_3]^b \times \ldots \times [y_n]^k$ where n is the number of dimensional variables and a, b, etc. are exponents.
4	Find the algebraic equations for the exponents that satisfy dimensional homogeneity (same power for dimensions on each side of equation).
5	Solve the equations for the exponents.
6	Express the dimensional equation in the form $y_1 = y_2^a y_3^b \ldots y_n^k$ and identify the π-groups.

EXAMPLE 8.3 π-GROUPS FOR DRAG ON A SPHERE USING EXPONENT METHOD

The drag of a sphere, F_D, in a flowing fluid is a function of the velocity V, the fluid density ρ, the fluid viscosity μ, and the sphere diameter D. Find the π-groups using the exponent method.

Problem Definition

Situation: Given $F_D = f(V, \rho, \mu, D)$.

Find: The π-groups using exponent method.

Plan

Follow the procedure for the exponent method in Table 8.2.

Solution

1. Dimensions of significant variables are

$$[F] = \frac{ML}{T^2},\ [V] = \frac{L}{T},\ [\rho] = \frac{M}{L^3},\ [\mu] = \frac{M}{LT},\ [D] = L$$

2. Number of π-groups is $5 - 3 = 2$.
3. Form product with dimensions.

$$\frac{ML}{T^2} = \left[\frac{L}{T}\right]^a \times \left[\frac{M}{L^3}\right]^b \times \left[\frac{M}{LT}\right]^c \times [L]^d$$

$$= \frac{L^{a-3b-c+d} M^{b+c}}{T^{a+c}}$$

4. Dimensional homogeneity. Equate powers of dimensions on each side.

$$L:\quad a - 3b - c + d = 1$$

$$M:\quad b + c = 1$$

$$T:\quad a + c = 2$$

5. Solve for exponents a, b, and c in terms of d.

$$\begin{pmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1-d \\ 1 \\ 2 \end{pmatrix}$$

The value of the determinant is –1 so a unique solution is achievable. Solution is $a = d, \quad b = d-1, \quad c = 2-d$

6. Write dimensional equation with exponents.

$$F = V^d \rho^{d-1} \mu^{2-d} D^d$$

$$F = \frac{\mu^2}{\rho}\left(\frac{\rho V D}{\mu}\right)^d$$

$$\frac{F\rho}{\mu^2} = \left(\frac{\rho V D}{\mu}\right)^d$$

There are two π-groups:

$$\pi_1 = \frac{F\rho}{\mu^2} \text{ and } \pi_2 = \frac{\rho VD}{\mu}$$

By dividing π_1 by the square of π_2, the π_1 group can be written as $F_D/(\rho V^2 D^2)$, so the functional form of the equation can be written as

$$\frac{F}{\rho V^2 D^2} = f\left(\frac{\rho VD}{\mu}\right)$$

8.4 Common π groups

Table 8.3 COMMON Π-GROUPS

π-Group	Symbol	Name	Ratio
$\frac{p - p_0}{(\rho V^2)/2}$	C_p	Pressure coefficient	$\frac{\text{Pressure differences}}{\text{Kinetic pressure}}$
$\frac{\tau}{(\rho V^2)/2}$	c_f	Shear-stress coefficient	$\frac{\text{Shear stress}}{\text{Kinetic pressure}}$
$\frac{F}{(\rho V^2 L^2)/2}$	C_F	Force coefficient	$\frac{\text{Force}}{\text{Kinetic force}}$

π-Group	Symbol	Name	Ratio
$\dfrac{\rho L V}{\mu}$	Re	Reynolds number	$\dfrac{\text{Kinetic force}}{\text{Viscous force}}$
$\dfrac{V}{c}$	M	Mach number	$\dfrac{\text{Kinetic force}}{\text{Compressive force}}$
$\dfrac{\rho L V^2}{\sigma}$	We	Weber number	$\dfrac{\text{Kinetic force}}{\text{Surface-tension force}}$
$\dfrac{V}{\sqrt{gL}}$	Fr	Froude number	$\dfrac{\text{Kinetic force}}{\text{Gravitational force}}$

8.5 Similitude

Similarity between model and prototype

Geometric similarity: the model must be the same shape as the prototype, but may be scaled by some constant scale factor

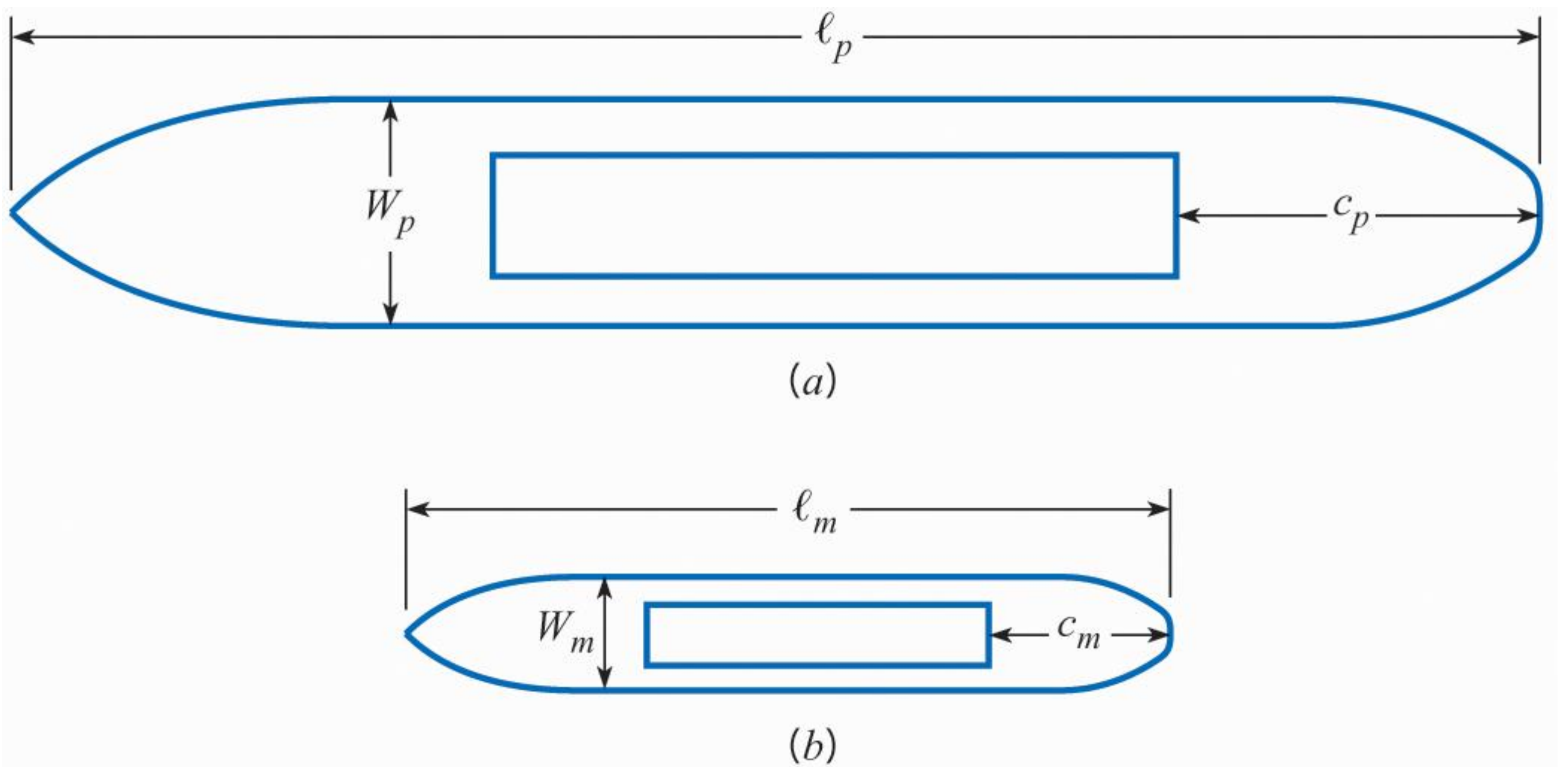
Kinematic similarity: velocity at any point in the model flow must be proportional (by a constant scale factor) to the velocity at the corresponding point in the prototype flow

Dynamic similarity: all forces in the model flow scale by a constant factor to corresponding forces in the prototype flow

In a general flow field, complete similarity between a model and prototype is achieved only when there is geometric, kinematic, and dynamic similarity.

Geometric similarity: the model must be the same shape as the prototype, but may be scaled by some constant scale factor

$$\frac{l_m}{l_p} = \frac{w_m}{w_p} = \frac{c_m}{c_p} = L_r \qquad A_r = L_r^2 \qquad \forall_r = L_r^3$$



Kinematic similarity: velocity at any point in the model flow must be proportional (by a constant scale factor) to the velocity at the corresponding point in the prototype flow

$$\frac{V_m}{V_p} = V_r = \frac{(L/t)_m}{(L/t)_p} = \frac{L_r}{t_r} \Rightarrow t_r = \frac{L_r}{V_r}$$

$$\frac{a_m}{a_p} = \frac{(V/t)_m}{(V/t)_p} = \frac{V_r}{t_r} = \frac{V_r}{(L_r/V_r)} = \frac{V_r^2}{L_r}$$

Dynamic similarity: all forces in the model flow scale by a constant factor to corresponding forces in the prototype flow

$$\frac{F_m}{F_p} = \frac{(ma)_m}{(ma)_p} = \frac{(\rho \forall V/t)_m}{(\rho \forall V/t)_p} = \frac{(\rho L^3 V/t)_m}{(\rho L^3 V/t)_p}$$

$$\frac{F_m}{F_p} = \frac{(\gamma L^3)_m}{(\gamma L^3)_p} = \frac{(\rho g L^3)_m}{(\rho g L^3)_p}$$

$$\frac{(\rho g L^3)_m}{(\rho g L^3)_p} = \frac{(\rho \forall V/t)_m}{(\rho \forall V/t)_p}, \quad \text{Also} \quad \frac{t_m}{t_p} = \frac{(L/V)_m}{(L/V)_p}$$

$$\left(\frac{V^2}{gL}\right)_m = \left(\frac{V^2}{gL}\right)_p \Rightarrow \left(\frac{V}{\sqrt{gL}}\right)_m = \left(\frac{V}{\sqrt{gL}}\right)_p \Rightarrow Fr_m = Fr_p$$

EXAMPLE 8.4 REYNOLDS-NUMBER SIMILITUDE

The drag characteristics of a blimp 5 m in diameter and 60 m long are to be studied in a wind tunnel. If the speed of the blimp through still air is 10 m/s, and if a 1/10 scale model is to be tested, what airspeed in the wind tunnel is needed for dynamically similar conditions? Assume the same air pressure and temperature for both model and prototype.

Problem Definition

Situation: Wind tunnel test of a 1/10 scale model blimp. Prototype speed is 10 m/s.

Find: Speed (in m/s) in wind tunnel for dynamic similitude.

Assumptions: Same air pressure and temperature for model and prototype, therefore $\nu_m = \nu_p$.

Plan

The only π-group that is appropriate is the Reynolds number (there are no compressibility effects, free-surface effects, or gravitation effects). Thus equating the model and prototype Reynolds number satisfies dynamic similitude.

1. Equate the Reynolds number of the model and the prototype.
2. Calculate model speed.

Solution

1. Reynolds-number similitude

$$\mathrm{Re}_m = \mathrm{Re}_p$$

$$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p}$$

2. Model velocity

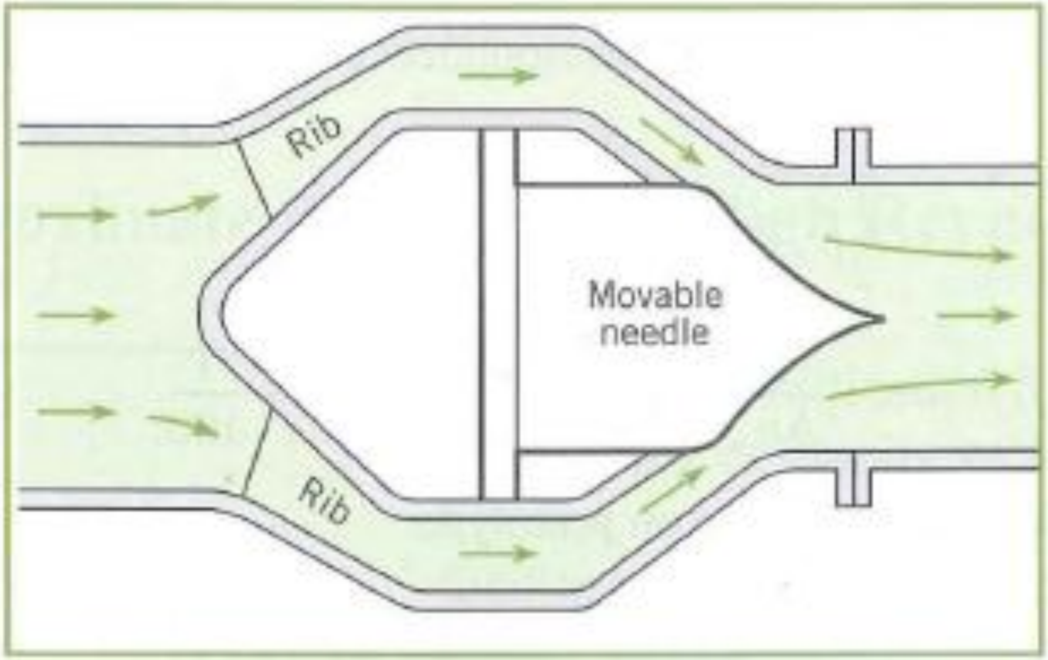
$$V_m = V_p \frac{L_p}{L_m} \frac{\nu_m}{\nu_p} = 10 \text{ m/s} \times 10 \times 1 = \boxed{100 \text{ m/s}}$$

EXAMPLE 8.5 REYNOLDS-NUMBER SIMILITUDE OF A VALVE

The valve shown is the type used in the control of water in large conduits. Model tests are to be done, using water as the fluid, to determine how the valve will operate under wide-open conditions. The prototype size is $\boxed{2\text{ m}}$ in diameter at the inlet. What flow rate is required for the model if the prototype flow is 20 m^3/s? Assume that the temperature for model and prototype is 15°C and that the model inlet diameter is 0.5 m.

3m

Sketch:



Solution

1. Reynolds-number similitude

$$\mathrm{Re}_m = \mathrm{Re}_p$$

$$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p}$$

2. Velocity ratio

$$\frac{V_m}{V_p} = \frac{L_p}{L_m}\frac{\nu_m}{\nu_p}$$

Since $\nu_p = \nu_m$,

$$\frac{V_m}{V_p} = \frac{L_p}{L_m}$$

3. Discharge

$$\frac{Q_m}{Q_p} = \frac{V_m A_m}{V_p A_p} = \frac{L_p}{L_m}\left(\frac{L_m}{L_p}\right)^2 = \frac{L_m}{L_p}$$

$$Q_m = 20\ \mathrm{m^3/s} \times \frac{1}{6} = \boxed{3.3\ \mathrm{m^3/s}}$$

EXAMPLE 8.7 DRAG FORCE FROM WIND TUNNEL TESTING

A 1/10 scale of a blimp is tested in a wind tunnel under dynamically similar conditions. If the drag force on the model blimp is measured to be 1530 N, what corresponding force could be expected on the prototype? The air pressure and temperature are the same for both model and prototype.

Problem Definition

Situation: A 1/10 scale model of blimp is tested in a wind tunnel, and a drag force of 1530 N is measured.

Find: The drag force (in newtons) on the prototype.

Properties: Pressure and temperature are the same, $\nu_m = \nu_p$.

Plan

Reynolds number is the only significant π-group, so Eq. (8.4) reduces to $C_F = f(\mathrm{Re})$. For dynamic similtude, $\mathrm{Re}_m = \mathrm{Re}_p$. Thus with geometric similitude $C_{F,m} = C_{F,p}$.

1. Find velocity ratio by equating Reynolds numbers.
2. Find the force ratio from force coefficient.

Solution

1. Reynolds-number similitude

$$\mathrm{Re}_m = \mathrm{Re}_p$$

$$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p}$$

$$\frac{V_p}{V_m} = \frac{L_m}{L_p} = \frac{1}{10}$$

2. Force coefficient correspondence

$$\frac{F_p}{\frac{1}{2}\rho_p V_p^2 L_p^2} = \frac{F_m}{\frac{1}{2}\rho_m V_m^2 L_m^2}$$

$$\frac{F_p}{F_m} = \frac{V_p^2}{V_m^2}\frac{L_p^2}{L_m^2} = \frac{L_m^2}{L_p^2}\frac{L_p^2}{L_m^2} = 1$$

$$F_p = 1530\text{N}$$