Chapter 7
The Energy equation

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Significant learning outcomes

Conceptual Knowledge

- Explain the meaning of energy, work, and power.
- Describe various type of head terms (pressure head, pump head, velocity head, turbine head, etc.).
- Explain the meaning of pump, turbine, efficiency, and head loss.
- List the steps used to derive the energy equation.

Procedural Knowledge

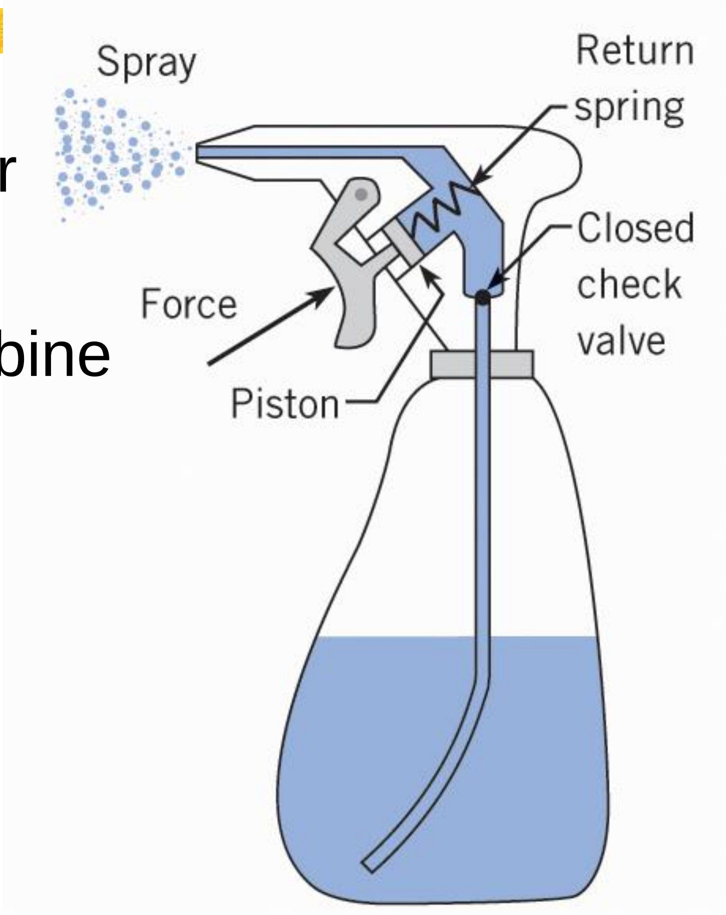
- Apply the energy equation to predict variables such as pressure drop and head loss.
- Apply the power equation to find the power required for a pump or power supplied by a turbine.
- Sketch an Energy Grade Line (EGL) or a Hydraulic Grade Line (HGL) and explain the trends.

Applications (typical)

- For a centrifugal pump or an axial fan, determine the requirements (e.g., energy, head, flow rate).
- For a turbine, establish how much power can be produced.
- For a piping system, identify locations of cavitation by sketching an HGL.

7.1 Energy, work and power

- **Definitions:** Energy, work and power
- **What is a machine?**
- **Hydraulic machines:** Pump and turbine



7.2 Energy Equation: General form

Total Energy of a system= Internal Energy + Kinetic Energy + Potential Energy

$$E = E_u + E_k + E_p$$

First law of thermodynamics: The change in the energy of a system is equal to the heat transferred to the system minus work done by the system

$$\Delta E = Q - W \qquad \frac{dE}{dt} = \dot{Q} - \dot{W}$$

$$B_{sys} = E \qquad b = e$$

$e =$ energy per unit mass

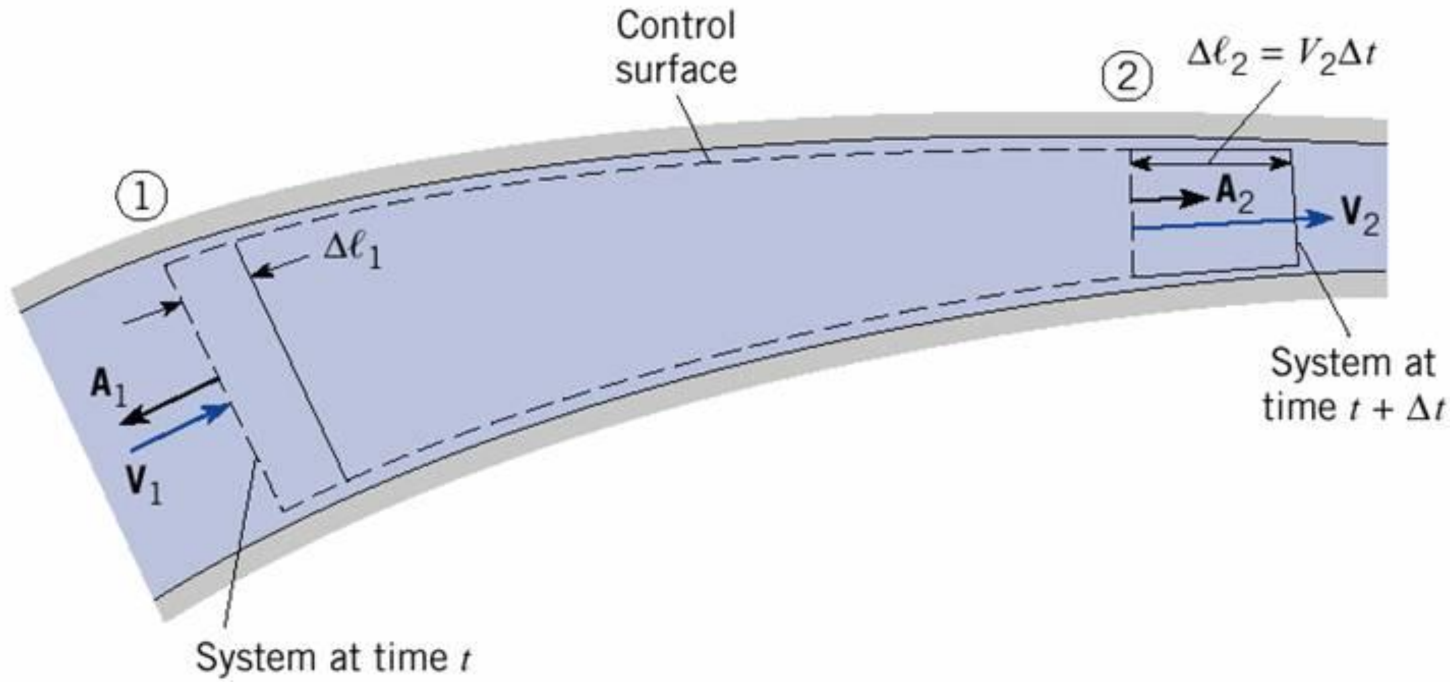
$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{cv} b\rho d\forall + \int_{cs} b\rho\vec{V}.d\vec{A}$$

$$\frac{dE_{sys}}{dt} = \frac{d}{dt}\int_{cv} e\rho d\forall + \int_{cs} e\rho\vec{V}.d\vec{A}$$

$$\dot{Q} - \dot{W} = \frac{d}{dt}\int_{cv} e\rho d\forall + \int_{cs} e\rho\vec{V}.d\vec{A}$$

$$e = e_k + e_p + u = \frac{V^2}{2} + gz + u$$

$$\dot{Q} - \dot{W} = \frac{d}{dt}\int_{cv}\left(\frac{V^2}{2} + gz + u\right)\rho d\forall + \int_{cs}\left(\frac{V^2}{2} + gz + u\right)\rho\vec{V}.d\vec{A}$$



Work is divided into **Flow Work** and **Shaft Work**

Upstream end: $\dot{W}_{f1} = p_1\vec{V}_1.\vec{A}_1 = -p_1V_1A_1$

Downstream end: $\dot{W}_{f2} = p_2\vec{V}_2.\vec{A}_2 = p_2V_2A_2$

Total flow work: $\boxed{\dot{W}_f = \sum_{cs}\frac{p}{\rho}\rho\vec{V}.\vec{A}}$

$$\dot{Q} - \dot{W}_s = \frac{d}{dt}\int_{cv}\left(\frac{V^2}{2} + gz + u\right)\rho d\forall + \int_{cs}\left(\frac{V^2}{2} + gz + u + \frac{p}{\rho}\right)\rho\vec{V}.d\vec{A}$$

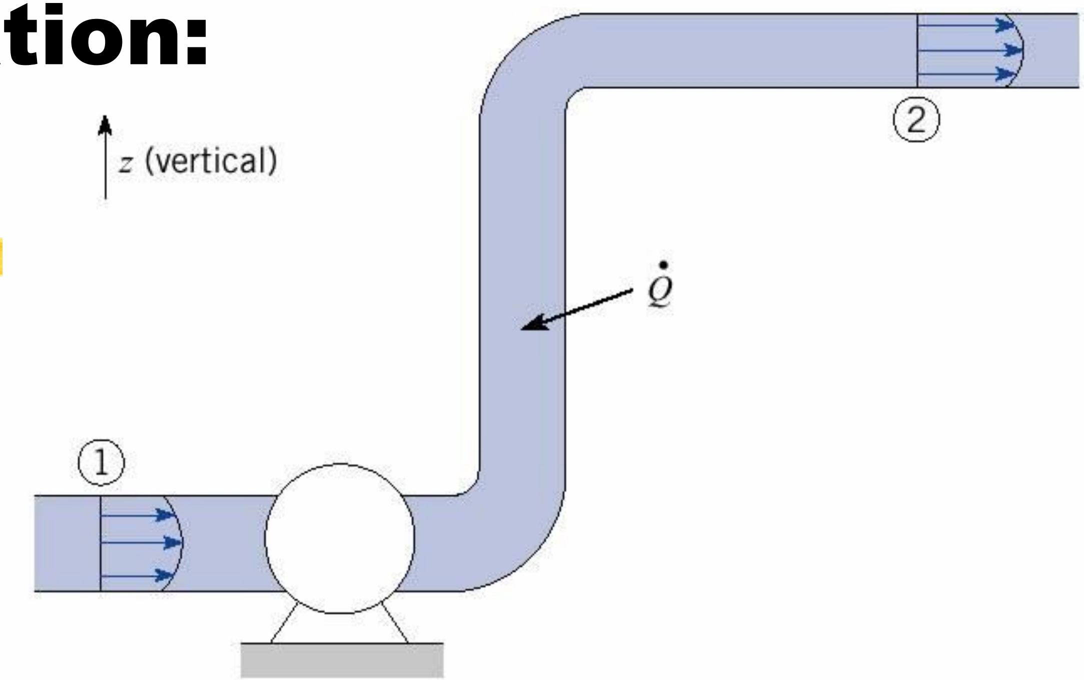
$$\dot{Q} - \dot{W}_s = \frac{d}{dt}\int_{cv}\left(\frac{V^2}{2} + gz + u\right)\rho d\forall + \int_{cs}\left(\frac{V^2}{2} + gz + h\right)\rho\vec{V}.d\vec{A}$$

Specific enthalpy: $h = \frac{p}{\rho} + u$

For steady flow

$$\dot{Q} - \dot{W}_s = \int_{cs}\left(\frac{V^2}{2} + gz + h\right)\rho\vec{V}.d\vec{A}$$

7.3 Energy equation: Pipe flow



For a steady flow, the energy equation between section 1 and 2 can be written as

$$\dot{Q} - \dot{W}_s + \int_{A_1} \left(\frac{p_1}{\rho} + g z_1 + u_1 \right) \rho V_1 dA_1 + \int_{A_1} \frac{\rho V_1^3}{2} dA_1$$

$$= \int_{A_2} \left(\frac{p_2}{\rho} + g z_2 + u_2 \right) \rho V_2 dA_2 + \int_{A_2} \frac{\rho V_2^3}{2} dA_2$$

At section 1 and 2, where the flow is uniform, p is constant over the cross-section, the internal energy is also constant across the sections, so

$$\dot{Q} - \dot{W}_s + \left(\frac{p_1}{\rho} + gz_1 + u_1\right)\int_{A_1} \rho V_1 dA_1 + \int_{A_1} \frac{\rho V_1^3}{2} dA_1$$
$$= \left(\frac{p_2}{\rho} + gz_2 + u_2\right)\int_{A_2} \rho V_2 dA_2 + \int_{A_2} \frac{\rho V_2^3}{2} dA_2$$

If average velocities at section 1 and 2 are denoted using over-bar on V and we use the following expressions

$$\int_A \frac{\rho V^3}{2} dA = \alpha \frac{\rho \bar{V}^3 A}{2}$$

$$\int_A \rho V dA = \rho \bar{V} A = \dot{m}$$

We can express the energy equation as

$$\frac{1}{\dot{m}g}\left(\dot{Q}-\dot{W}_s\right)+\frac{p_1}{\gamma}+z_1+\alpha_1\frac{\bar{V}_1^{\,2}}{2g}=\frac{p_2}{\gamma}+z_2+\alpha_2\frac{\bar{V}_2^{\,2}}{2g}+\frac{u_2-u_1}{g}$$

The factor α is called as kinetic energy correction factor. It is equal to unity for uniform velocity across the section and greater than 1 for non uniform velocity distribution (which is the case in most of the practical situations).

$$\alpha=\frac{1}{A}\int_A\left(\frac{V}{\bar{V}}\right)^3 dA$$

The value of α for laminar flow is 2.0 which can be proved from the laminar velocity profile

The shaft-work term in the above equations is usually the result of a turbine or pump in the flow network. Therefore

$$\dot{W}_s = \dot{W}_t - \dot{W}_p$$

And we can rearrange the above energy equation as

$$\frac{\dot{W}_p}{\dot{m}g} + \frac{p_1}{\gamma} + z_1 + \alpha_1 \frac{\overline{V}_1^{\,2}}{2g} = \frac{\dot{W}_t}{\dot{m}g} + \frac{p_2}{\gamma} + z_2 + \alpha_2 \frac{\overline{V}_2^{\,2}}{2g} + \frac{u_2 - u_1}{g} - \frac{\dot{Q}}{\dot{m}g}$$

$\dfrac{\dot{W}_p}{\dot{m}g} = h_p$:Head supplied by the pump to the fluid

$\dfrac{\dot{W}_t}{\dot{m}g} = h_t$:Head used by the turbine from the fluid

$$\left[\frac{u_2 - u_1}{g} - \frac{\dot{Q}}{\dot{m}g}\right] = h_L$$:Head loss

Energy equation can be written in a general form for steady flow

$$\frac{p_1}{\gamma} + z_1 + \alpha_1 \frac{\overline{V}_1^{\,2}}{2g} + h_p = \frac{p_2}{\gamma} + z_2 + \alpha_2 \frac{\overline{V}_2^{\,2}}{2g} + h_t + h_L$$

- **Special Case** : Incompressible flow with no mechanical work devices and negligible friction

$$\frac{p_1}{\gamma} + z_1 + \alpha_1 \frac{\overline{V}_1^{\,2}}{2g} = \frac{p_2}{\gamma} + z_2 + \alpha_2 \frac{\overline{V}_2^{\,2}}{2g}$$

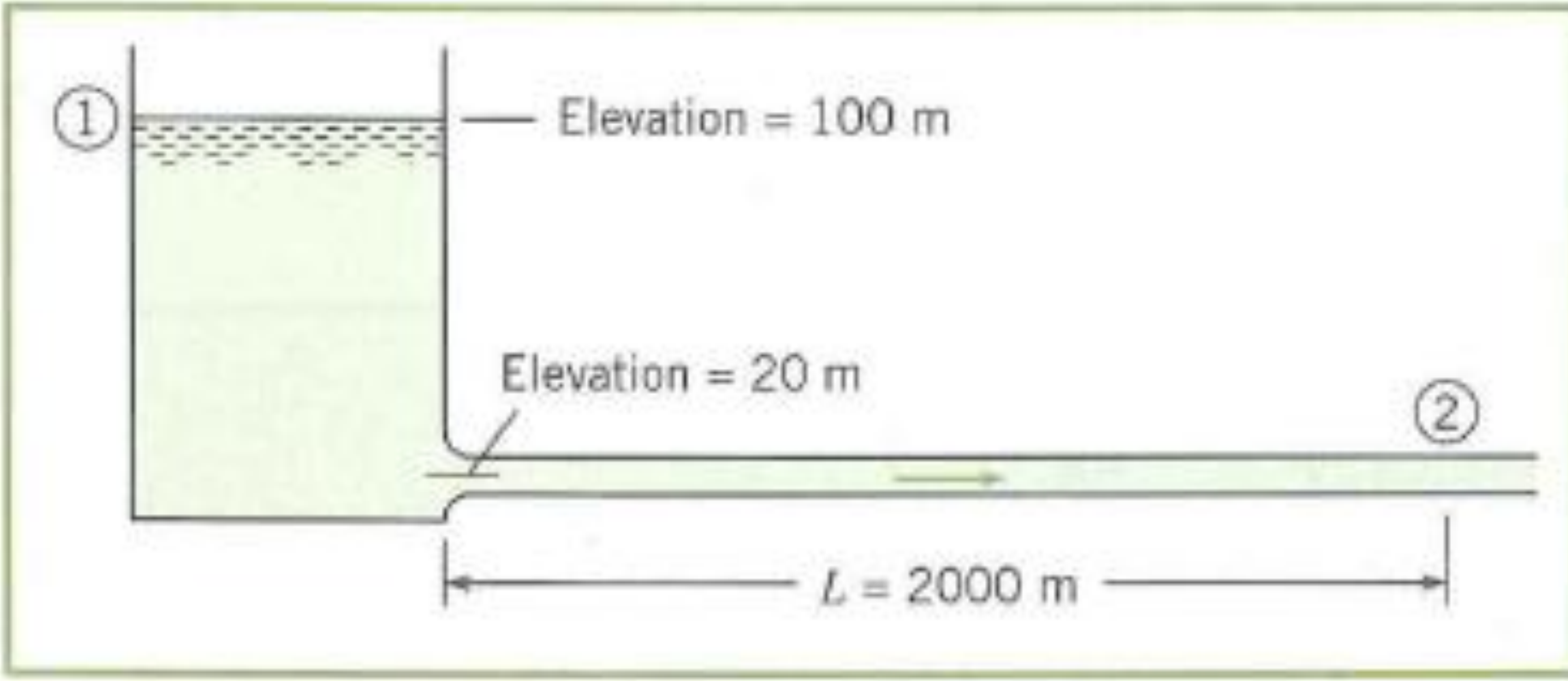
We get the well known Bernoulli Equation.

EXAMPLE 7.2 PRESSURE IN A PIPE

A horizontal pipe carries cooling water at 10°C for a thermal power plant from a reservoir as shown. The head loss in the pipe is

$$h_L = \frac{0.02(L/D)V^2}{2g}$$

where L is the length of the pipe from the reservoir to the point in question, V is the mean velocity in the pipe, and D is the diameter of the pipe. If the pipe diameter is 20 cm and the rate of flow is 0.06 m^3/s, what is the pressure in the pipe at $L = 2000$ m. Assume $\alpha_2 = 1$.



Assumptions: $\alpha_2 = 1.0$.

Properties: Water (10°C), Table A.5: $\gamma = 9810\ \text{N/m}^3$.

Solution

1. Energy equation (general form)

$$\frac{p_1}{\gamma} + \alpha_1 \frac{\overline{V}_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2 \frac{\overline{V}_2^2}{2g} + z_2 + h_t + h_L$$

2. Term-by-term analysis
   - $p_1 = 0$ because the pressure at top of a reservoir is $p_{atm} = 0$ gage.
   - $V_1 \approx 0$ because the level of the reservoir is constant or changing very slowly.
   - $z_1 = 100$ m; $z_2 = 20$ m.
   - $h_p = h_t = 0$ because there are no pumps or turbines in the system.
   - Find V_2 using the flow rate equation from Eq. (5.3).

$$V_2 = \frac{Q}{A} = \frac{0.06 \text{ m}^3/\text{s}}{(\pi/4)(0.2 \text{ m})^2} = 1.910 \text{ m/s}$$

- Head loss is

$$h_L = \frac{0.02(L/D)V^2}{2g} = \frac{0.02(2000\text{ m}/0.2\text{ m})(1.910\text{ m/s})^2}{2(9.81\text{ m/s}^2)}$$

$$= 37.2\text{ m}$$

3. Combine steps 1 and 2.

$$(z_1 - z_2) = \frac{p_2}{\gamma} + \alpha_2 \frac{\overline{V}_2^2}{2g} + h_L$$

$$80\text{ m} = \frac{p_2}{\gamma} + 1.0\frac{(1.910\text{ m/s})^2}{2(9.81\text{ m/s}^2)} + 37.2\text{ m}$$

$$80\text{ m} = \frac{p_2}{\gamma} + (0.186\text{ m}) + (37.2\text{ m})$$

$$p_2 = \gamma(42.6\text{ m}) = (9810\text{ N/m}^3)(42.6\text{ m}) = \boxed{418\text{ kPa}}$$

7.4 Power equation

$$\dot{W}_p = \dot{m}gh_p = \gamma Q h_p$$:Power supplied by the pump to the fluid

$$\dot{W}_t = \dot{m}gh_t = \gamma Q h_t$$:Power supplied to the turbine by the fluid

η_p :Mechanical efficiency of the pump

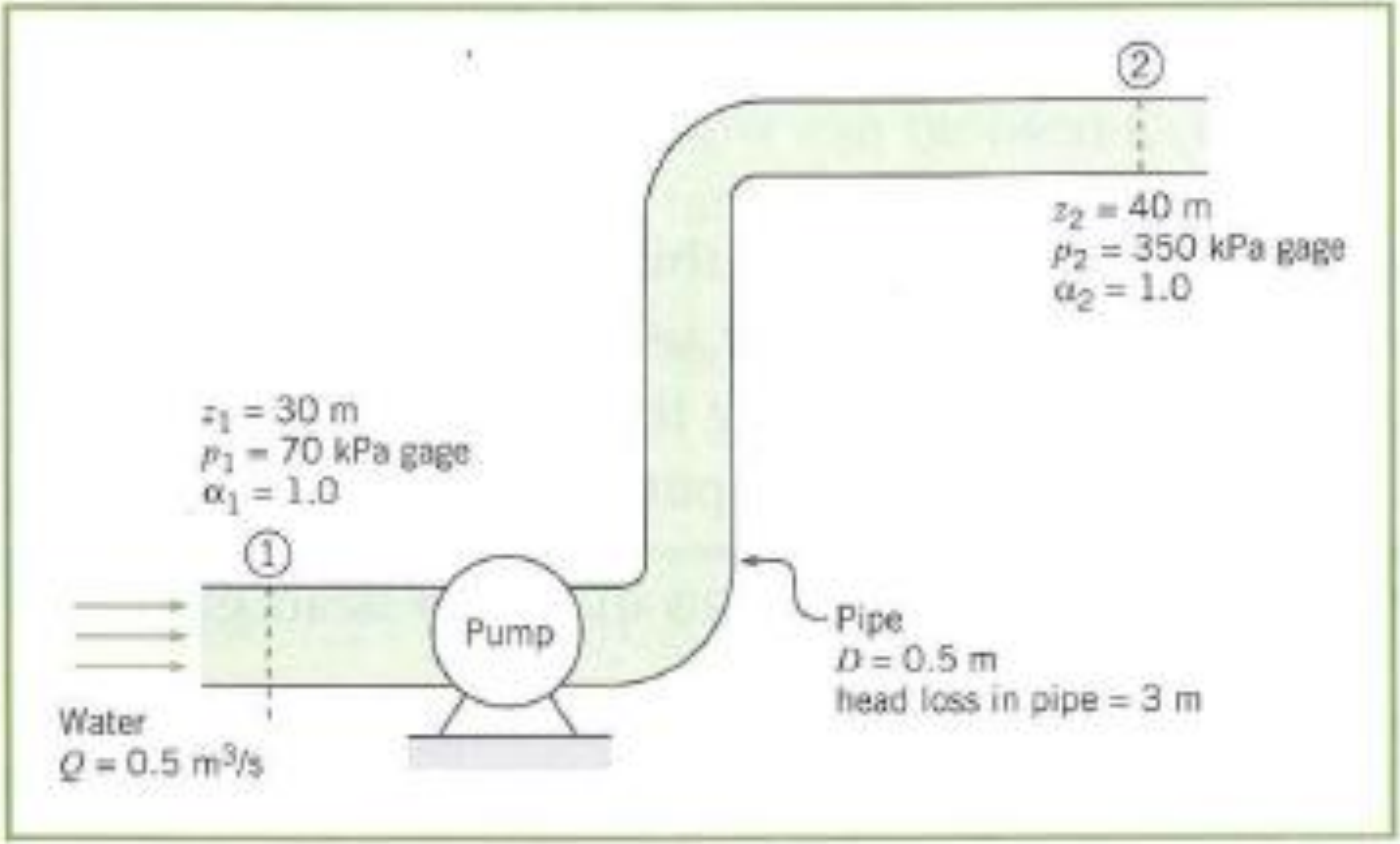
η_t :Mechanical efficiency of the turbine

Power input to the pump by motor: $\dfrac{\dot{W}_p}{\eta_p}$

Power output from the turbine to generator: $\eta_t \dot{W}_t$

EXAMPLE 7.3 POWER NEEDED BY A PUMP

A pipe 50 cm in diameter carries water (10°C) at a rate of $0.5\ m^3/s$. A pump in the pipe is used to move the water from an elevation of 30 m to 40 m. The pressure at section 1 is 70 kPa gage and the pressure at section 2 is 350 kPa gage. What power in kilowatts and in horsepower must be supplied to the flow by the pump? Assume $h_L = 3$ m of water and $\alpha_1 = \alpha_2 = 1$.



2
$z_2 = 40$ m
$p_2 = 350$ kPa gage
$\alpha_2 = 1.0$
$z_1 = 30$ m
$p_1 = 70$ kPa gage
$\alpha_1 = 1.0$
1
Pump
Pipe
$D = 0.5$ m
head loss in pipe = 3 m
Water
$Q = 0.5\ m^3/s$

Solution

1. Energy equation (general form)

$$\frac{p_1}{\gamma} + \alpha_1 \frac{\overline{V}_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2 \frac{\overline{V}_2^2}{2g} + z_2 + h_t + h_L$$

2. Term-by-term analysis
   - Velocity head cancels because $V_1 = V_2$.
   - $h_t = 0$ because there are no turbines in the system.
   - All other head terms are given.
   - Inserting terms into the general equation gives

$$\frac{p_1}{\gamma} + z_1 + h_p = \frac{p_2}{\gamma} + z_2 + h_L$$

3. Pump head (from step 2)

$$h_p = \left(\frac{p_2 - p_1}{\gamma}\right) + (z_2 - z_1) + h_L$$

$$= \left(\frac{(350{,}000 - 70{,}000)\ \text{N/m}^2}{9810\ \text{N/m}^3}\right) + (10\ \text{m}) + (3\ \text{m})$$

$$= (28.5\ \text{m}) + (10\ \text{m}) + (3\ \text{m}) = 41.5\ \text{m}$$

Interpretation: The head provided by the pump (41.5 m) is balanced by the increase in pressure head (28.5 m) plus the increase in elevation head (10 m) plus the head loss (3 m).

4. Power equation

$$P = \gamma Q h_p$$

$$= (9810\ \text{N/m}^3)(0.5\ \text{m}^3/\text{s})(41.5\ \text{m})$$

$$= \boxed{204\ \text{kW}}$$

EXAMPLE 7.4 POWER PRODUCED BY A TURBINE

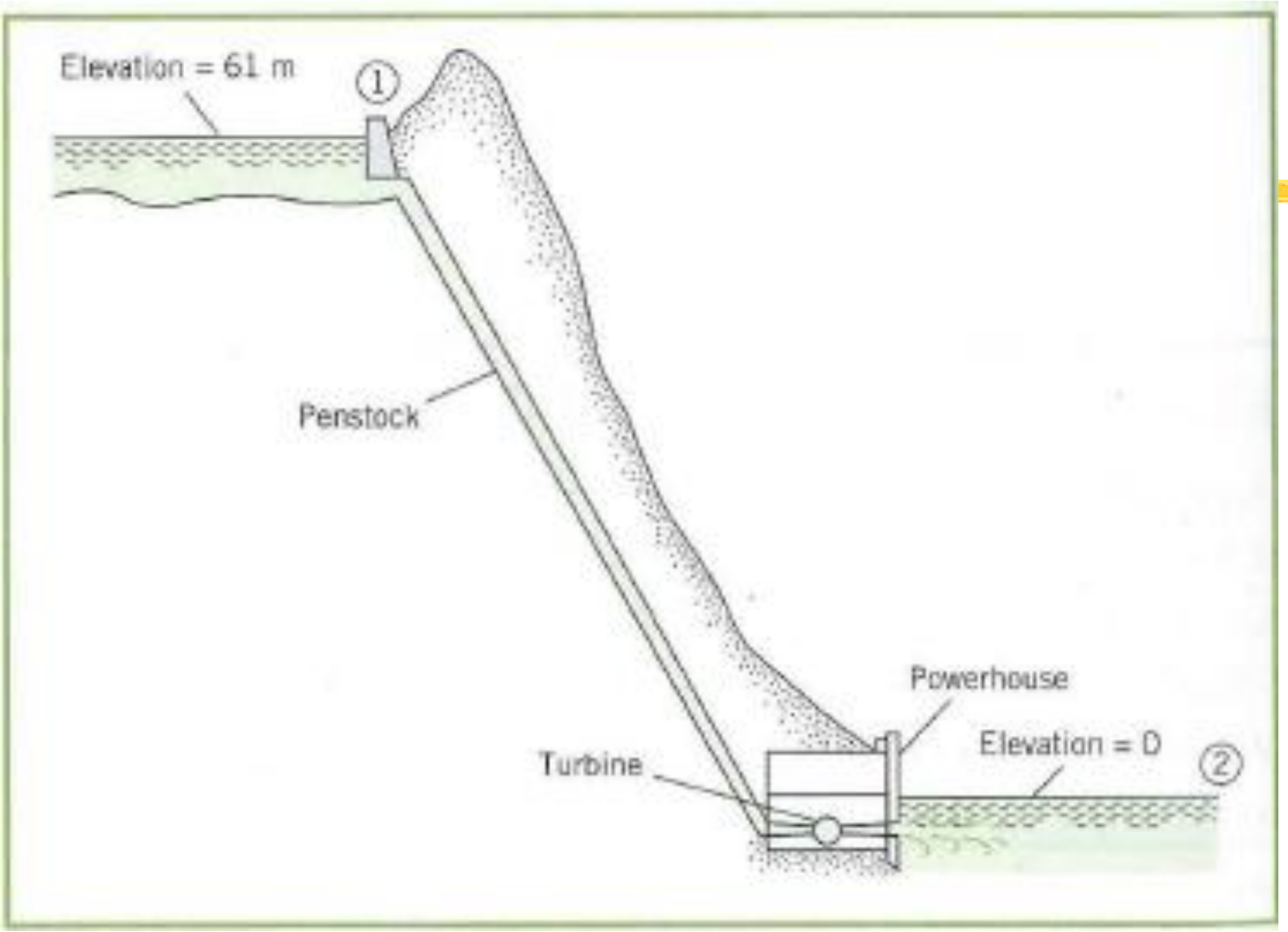
At the maximum rate of power generation, a small hydroelectric power plant takes a discharge of 14.1 m^3/s through an elevation drop of 61 m. The head loss through the intakes, penstock, and outlet works is 1.5 m. The combined efficiency of the turbine and electrical generator is 87%. What is the rate of power generation?

Problem Definition

Situation: A small hydroelectric plant is producing electrical power.

Find: Electrical power generation (in kW).

Properties: Water (10°C), Table A.5: $\gamma = 9810\ N/m^3$.


Elevation = 61 m
1
Penstock
Powerhouse
Elevation = 0
2
Turbine

Plan

1. Write the energy equation (7.29) between section 1 and section 2.
2. Analyze each term in the energy equation.
3. Solve for the head of the turbine h_t.
4. Find the input power to the turbine using the power equation (7.30a).
5. Find the output power from generator by using the efficiency equation (7.32).

Solution

1. Energy equation (general form)

$$\frac{p_1}{\gamma} + \alpha_1 \frac{\overline{V}_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2 \frac{\overline{V}_2^2}{2g} + z_2 + h_t + h_L$$

2. Term-by-term analysis
   - Velocity heads are negligible because $V_1 \approx 0$ and $V_2 \approx 0$.
   - Pressure heads are zero because $p_1 = p_2 = 0$ gage.

- $h_p = 0$ because there is no pump in the system.
- Elevation head terms are given.

3. Combine steps 1 and 2:

$$h_t = (z_1 - z_2) - h_L$$
$$= (61\text{ m}) - (1.5\text{ m}) = 59.5\text{ m}$$

Interpretation: Head supplied to the turbine (59.5 m) is equal to the net elevation change of the dam (61 m) minus the head loss (1.5 m).

4. Power equation

$$P_{\text{input to turbine}} = \gamma Q h_t = (9810\text{ N/m}^3)(14.1\text{ m}^3/\text{s})(59.5\text{ m})$$
$$= 8.23\text{ MW}$$

5. Efficiency equation

$$P_{\text{output from generator}} = \eta P_{\text{input to turbine}} = 0.87(8.23\text{ MW})$$
$$= \boxed{7.16\text{ MW}}$$

7.7 Hydraulic Grade Line (HGL) and Energy Grade Line (EGL)

$$\frac{p_1}{\gamma}+z_1+\alpha_1\frac{V_1^2}{2g}+h_p=\frac{p_2}{\gamma}+z_2+\alpha_2\frac{V_2^2}{2g}+h_t+h_L$$

Tips for Drawing HGLs and EGLs

1. In a lake or reservoir, the HGL and EGL will coincide with the liquid surface. Also, both the HGL and EGL will indicate piezometric head (Fig.7.7).

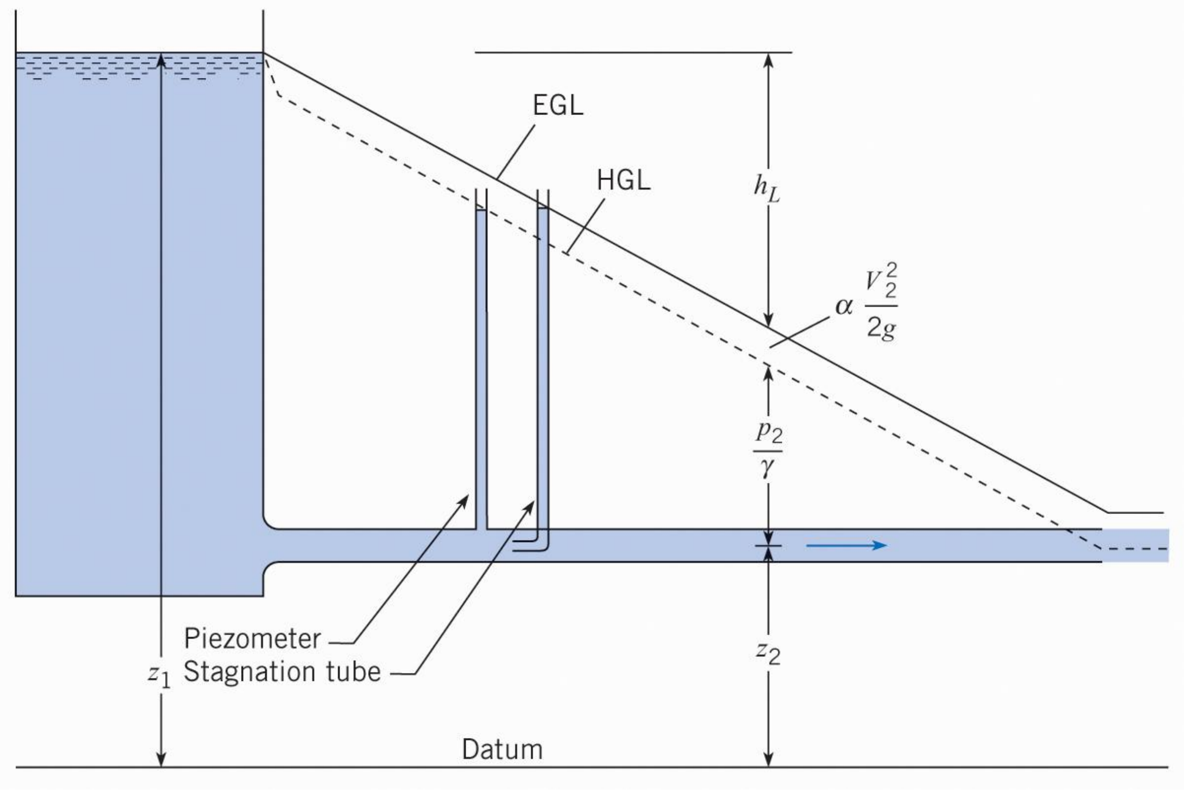


Fig. 7.7

2. A pump causes an abrupt rise in the EGL and HGL by adding energy to the flow. (Fig.7.8).

3. For steady flow in a Pipe of constant diameter and wall roughness, the slope of the EGL and the HGL will be constant (Fig. 7.7).

4. Locate the HGL below the EGL by a distance of the velocity head ($\alpha V^2/2g$).

5. Height of the EGL decreases in the flow direction unless a pump is present.

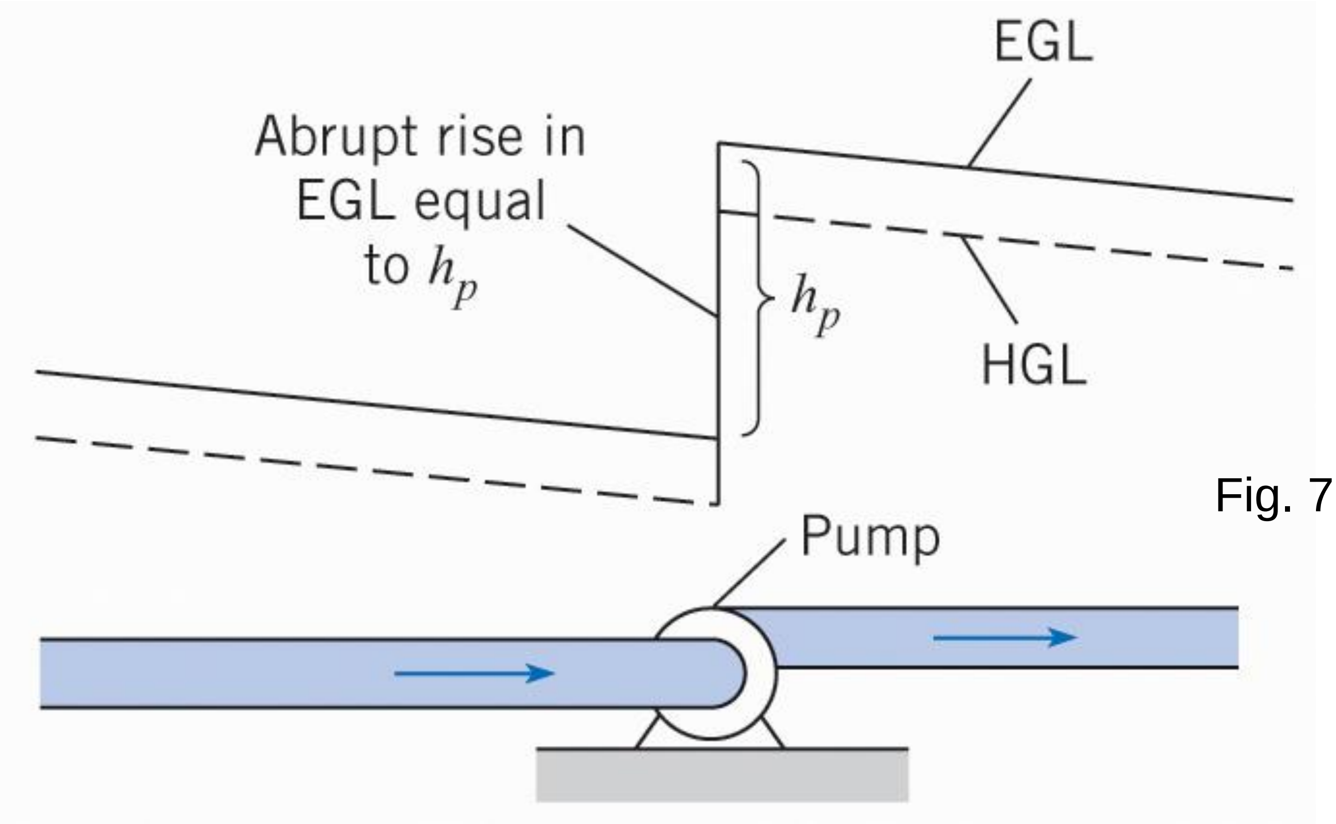


Fig. 7.8

6. A turbine causes an abrupt drop in the EGL and HGL by removing energy from the flow (Fig. 7.9).

7. Power generated by a turbine can be increased by using a gradual expansion at the turbine outlet. As shown in Fig. 7.9, the expansion converts kinetic energy to pressure. If the outlet to a reservoir is an abrupt expansion, as in Fig. 7.11, this kinetic energy is lost.

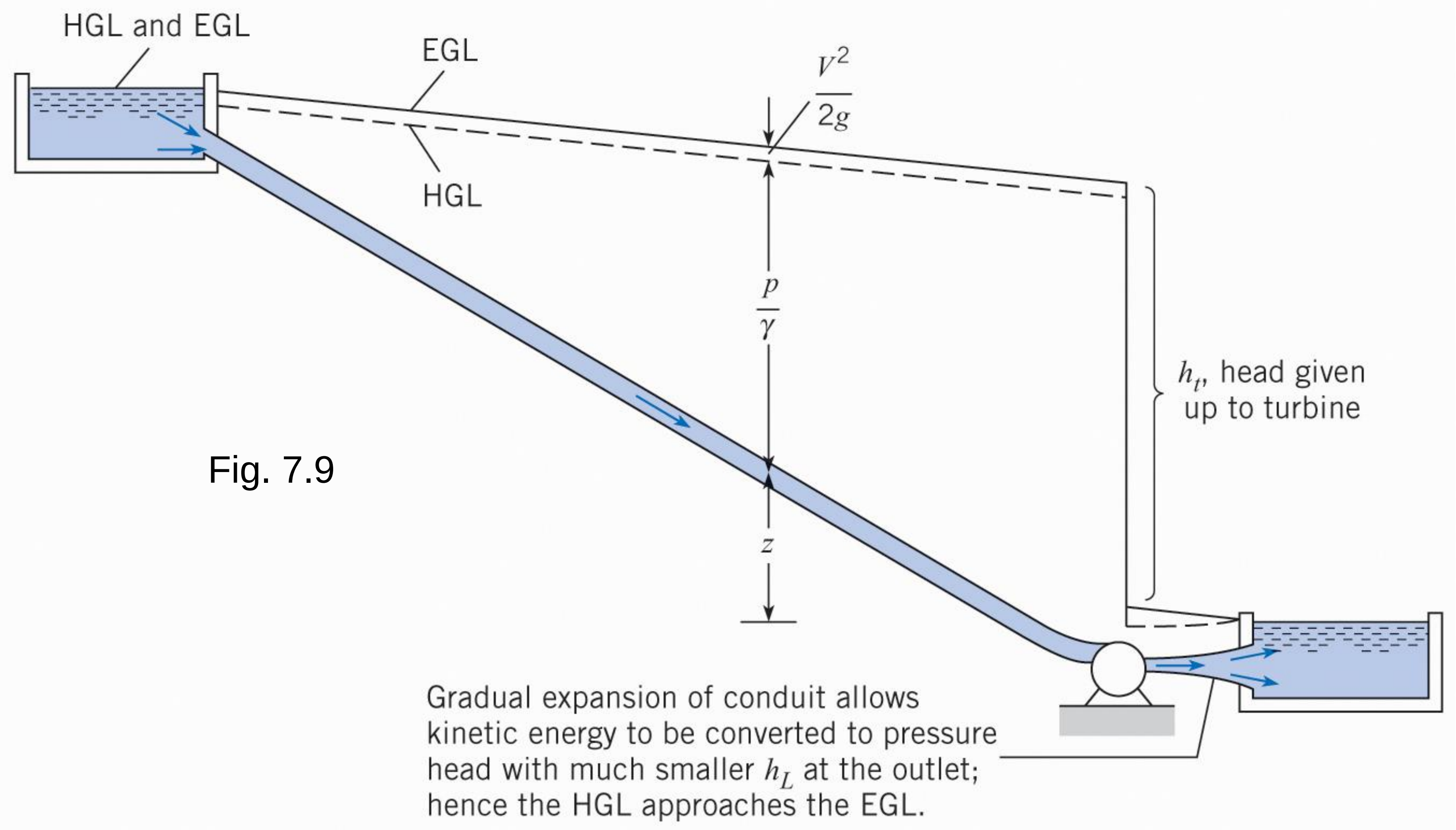


Fig. 7.9

8. When a pipe discharges into the atmosphere the HGL is coincident with the system because $p/\gamma = 0$ at these points. For example, in Figures 7.10 and 7.12, the HGL in the liquid jet is drawn through the jet itself.

9. When a flow passage changes diameter, the distance between the EGL and the HGL will change (Fig. 7.10 and Fig. 7.11) because velocity changes. In addition, the slope on the EGL will change because the head loss per length will be larger in the conduit with the larger velocity (see Fig. 7.11).

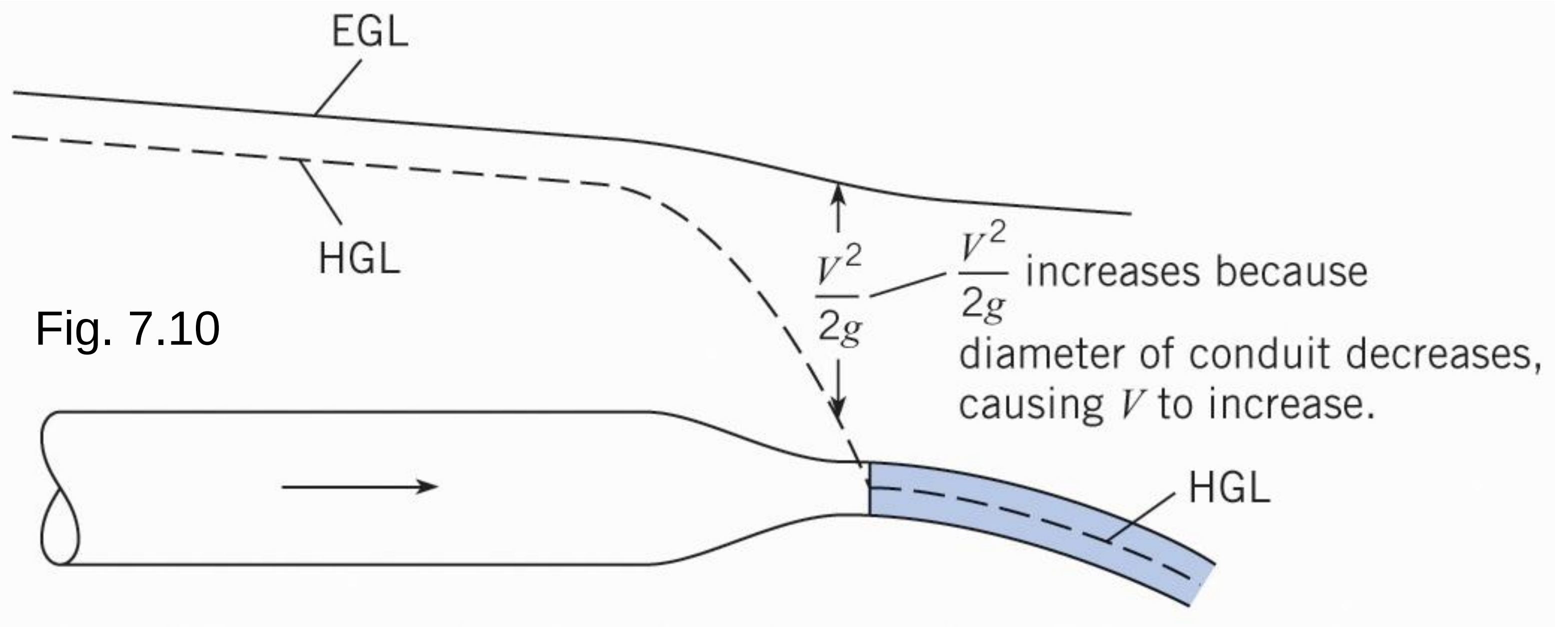


Fig. 7.10

10. If the HGL falls below the pipe, then p/γ is negative, indicating sub-atmospheric pressure (Fig. 7.12) and a potential location of cavitation.

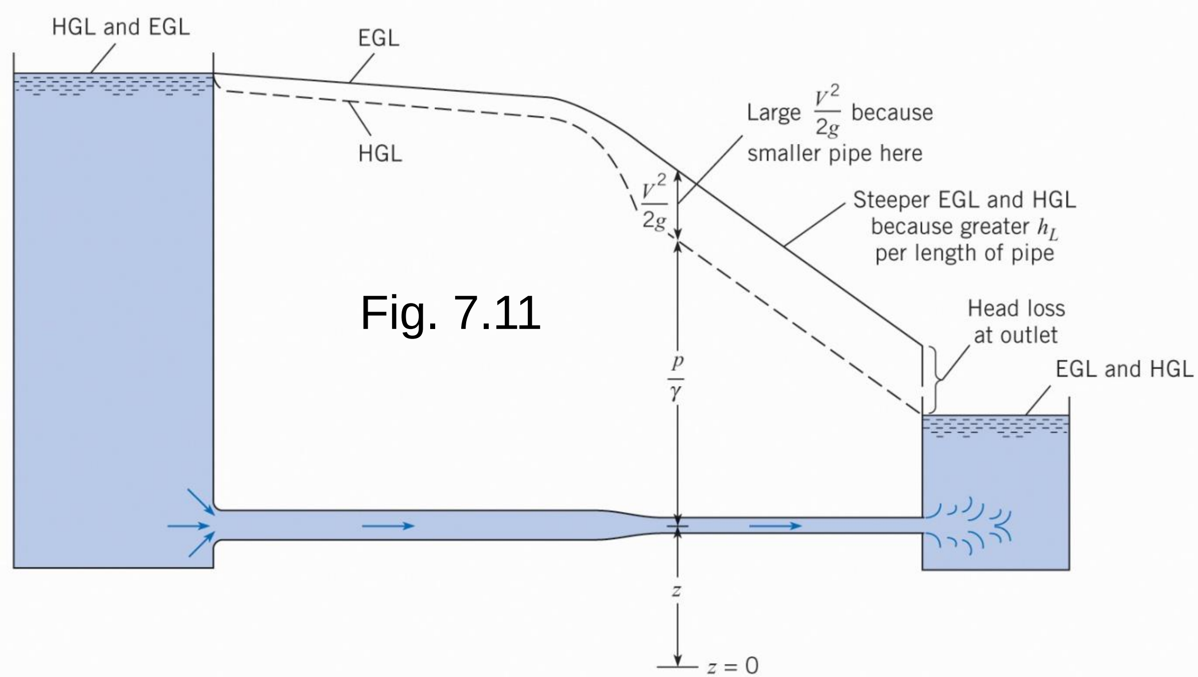


Fig. 7.11

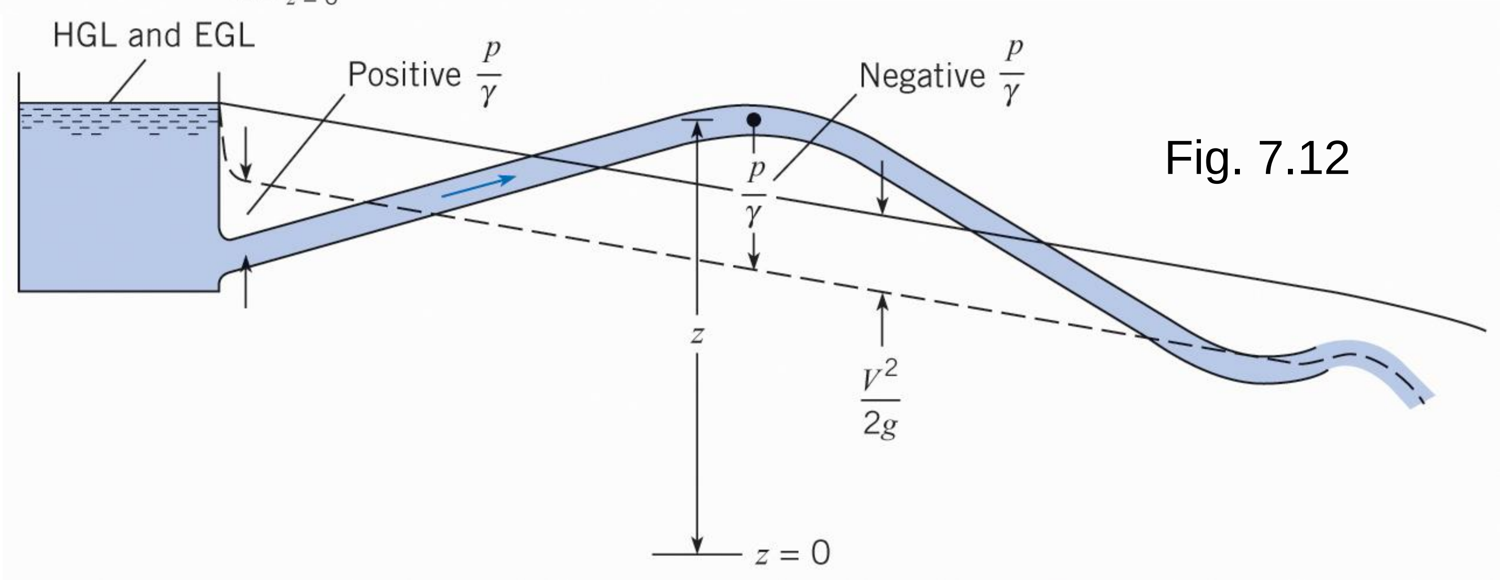


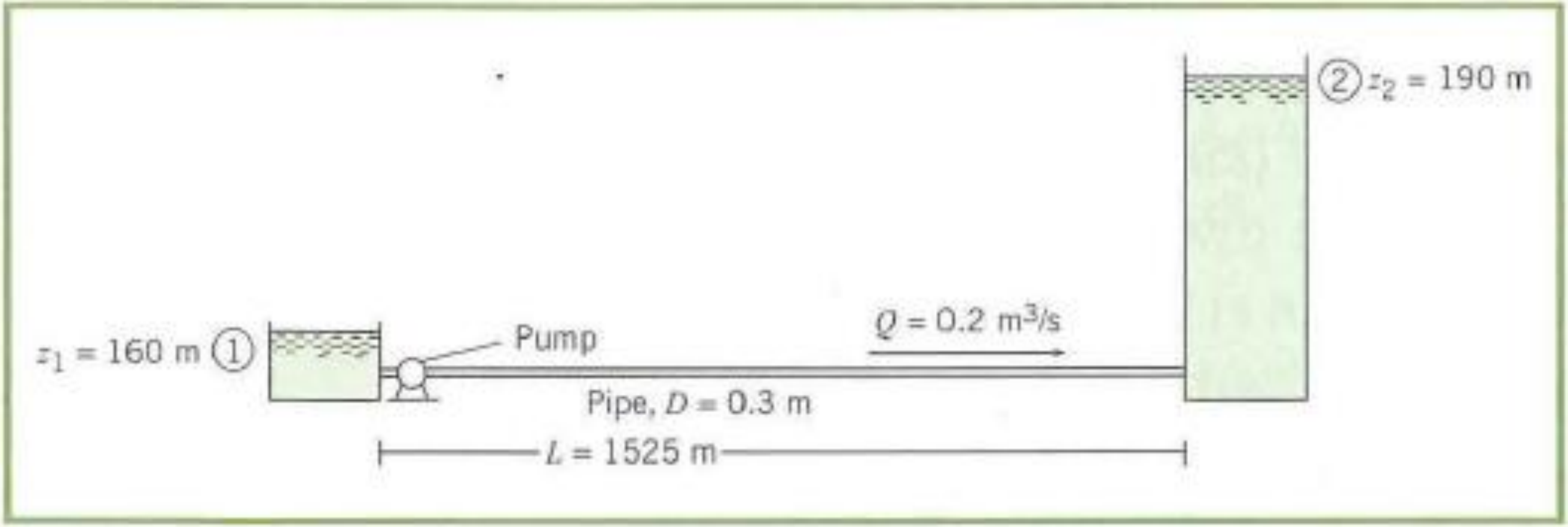
Fig. 7.12

EXAMPLE 7.6 EGL AND HGL FOR A SYSTEM

A pump draws water (10°C) from a reservoir, where the water-surface elevation is 160 m, and forces the water through a pipe 1525 m long and 0.3 m in diameter. This pipe then discharges the water into a reservoir with water-surface elevation of 190 m. The flow rate is 0.2 m^3/s, and the head loss in the pipe is given by

$$h_L = 0.01\left(\frac{L}{D}\right)\left(\frac{V^2}{2g}\right)$$

Determine the head supplied by the pump, h_p, and the power supplied to the flow, and draw the HGL and EGL for the system. Assume that the pipe is horizontal and is 155 m in elevation.



z_2 = 190 m
2
z_1 = 160 m
1
Pump
Q = 0.2 m³/s
Pipe, D = 0.3 m
L = 1525 m

Solution

1. Energy equation (general form)

$$\frac{p_1}{\gamma} + \alpha_1 \frac{\overline{V}_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2 \frac{\overline{V}_2^2}{2g} + z_2 + h_t + h_L$$

- Velocity heads are negligible because $V_1 \approx 0$ and $V_2 \approx 0$.
- Pressure heads are zero because $p_1 = p_2 = 0$ gage.
- $h_t = 0$ because there are no turbines in the system.

$$h_p = (z_2 - z_1) + h_L$$

Interpretation: Head supplied by the pump provides the energy to lift the fluid to a higher elevation plus the energy to overcome head loss.

2. Calculations of terms in the energy equation

- Calculate V using the flow rate equation.

$$V = \frac{Q}{A} = \frac{0.2\ \text{m}^3/\text{s}}{(\pi/4)(0.3\ \text{m})^2} = 2.83\ \text{m/s}$$

- Calculate head loss.

$$h_L = 0.01\left(\frac{L}{D}\right)\left(\frac{V^2}{2g}\right) = 0.01\left(\frac{1525 \text{ m}}{0.3 \text{ m}}\right)\left(\frac{(2.83 \text{ m/s})^2}{2 \times (9.81 \text{ m/s}^2)}\right)$$

$$= 20.75 \text{ m}$$

- Calculate h_p.

$$h_p = (z_2 - z_1) + h_L = (190 \text{ m} - 160 \text{ m}) + 20.75 \text{ m} = \boxed{50.75 \text{ m}}$$

3. Power

$$\dot{W}_p = \gamma Q h_p = \left(\frac{9810 \cancel{\text{N}}}{\cancel{\text{m}}^3}\right)\left(\frac{0.2 \cancel{\text{m}}^3}{\cancel{\text{s}}}\right)(50.75 \cancel{\text{m}})\left(\frac{\text{W} \cdot \cancel{\text{s}}}{\cancel{\text{N}} \cdot \cancel{\text{m}}}\right)$$

$$= \boxed{99.5 \text{ kW}}$$

4. HGL and EGL

- From Tip 1 on p. 233, locate the HGL and EGL along the reservoir surfaces.
- From Tip 2, sketch in a head rise of 50.75 m corresponding to the pump.

- From Tip 3, sketch the EGL from the pump outlet to the reservoir surface. Use the fact that the head loss is 77.6 ft. Also, sketch EGL from the reservoir on the left to the pump inlet. Show a small head loss.
- From Tip 4, sketch the HGL below the EGL by a distance of $V^2/2g \approx 0.4$ m.
- From Tip 5, check the sketches to ensure that EGL and HGL are decreasing in the direction of flow (except at the pump).

Sketch: HGL (dashed black line) and EGL (solid blue line)