# Chapter 5
# Control volume approach and continuity equation

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# Significant learning outcomes

## Conceptual Knowledge

- Explain the key difference between the Lagrangian and Eulerian descriptions of a flow field.
- Explain the meaning of volume flow rate and mass flow rate.
- Explain what is meant by a system, control volume and control surface.
- State the purpose of the Reynolds transport theorem.
- Outline the steps in the derivation of the continuity equation.

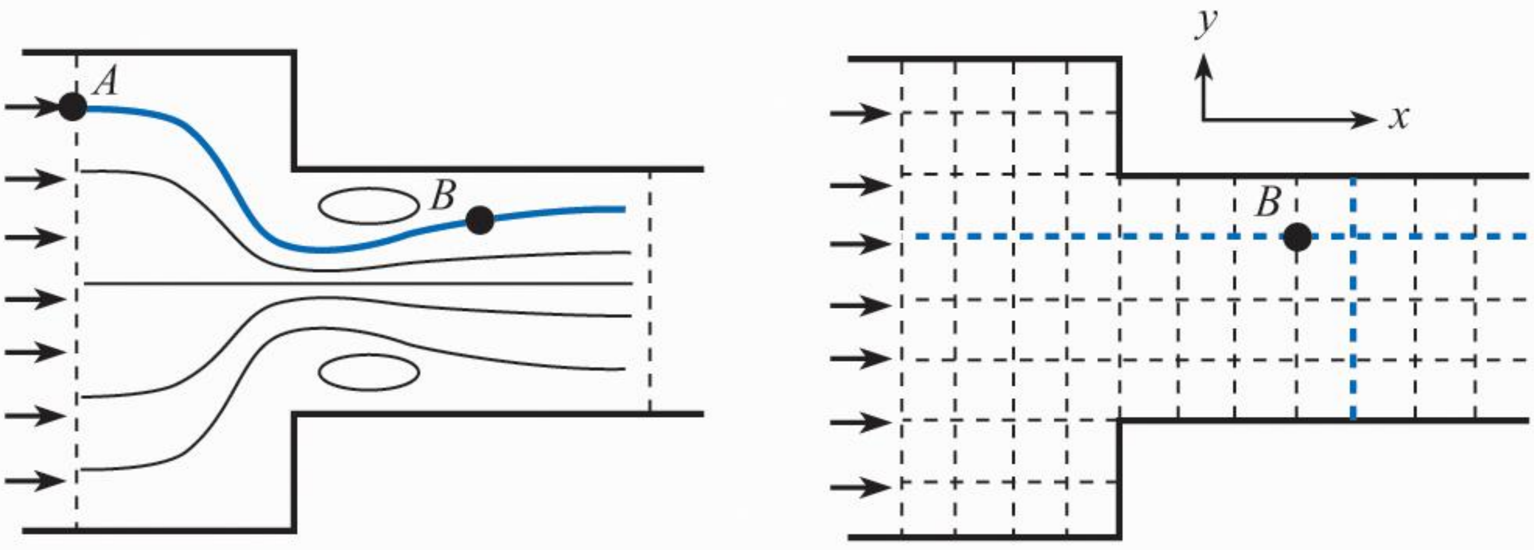
## Procedural Knowledge

- Calculate the volume flow rate (discharge) and the mass flow rate.
- Apply the continuity equation to draining tanks and reservoirs.
- Apply continuity equation to velocity changes in variable-area ducts.

## Applications (typical)

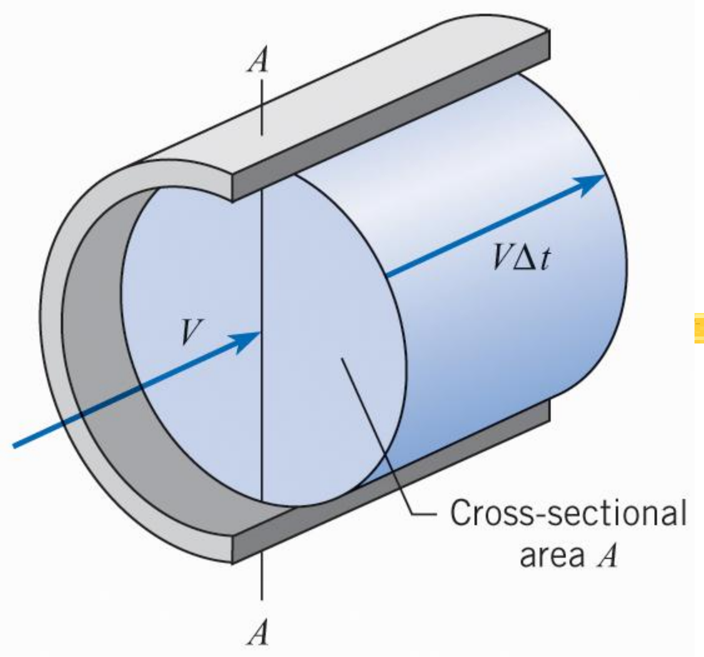
- For flow through a Venturimeter, relate pressure, velocity, and flow rate.
- For a tank or reservoir, estimate draining time.
- For a small leak in a pressurized chamber, estimate depressurization time.

# 5.1 Lagrangian and Eulerian approach




Lagrangian approach Eulerian approach

# 5.1 Rate of flow

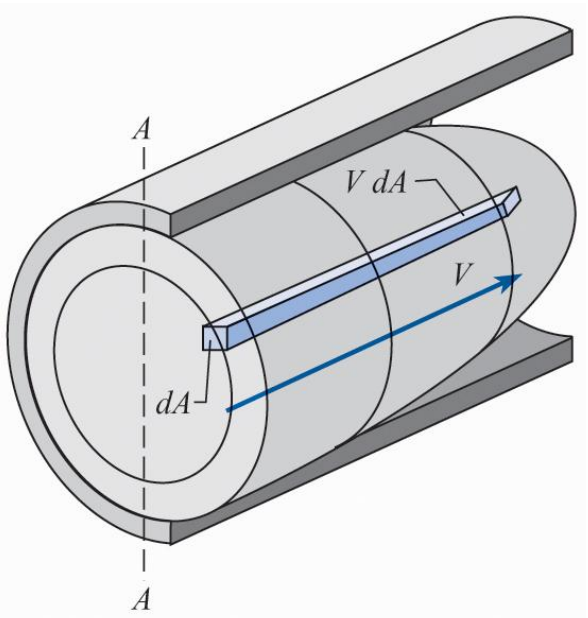



**Discharge (volume flow rate):**
Volume of fluid passing through an area per unit time.

$$Q = \lim_{\Delta t \to 0} \frac{\Delta \forall}{\Delta t} = VA$$

$$Q = \int_A V dA$$

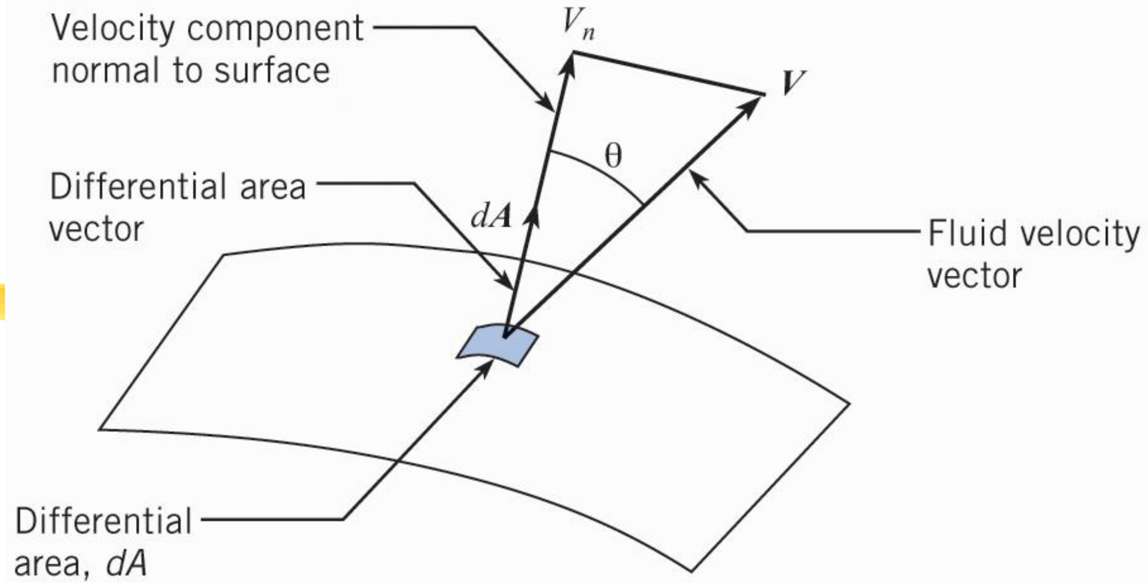
$$\bar{V} = \frac{Q}{A}$$



$$Q = \int_A V_n dA$$

$$= \int_A |V| \cos\theta dA$$

$$Q = \int_A \vec{V}.d\vec{A}$$




**Mass flow rate**: Mass of fluid passing through an area per unit time.

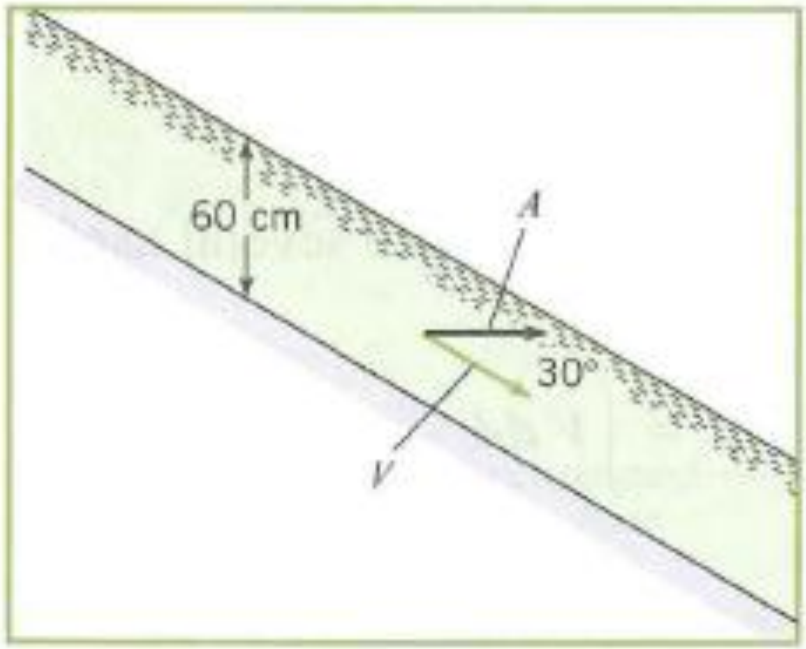
$$\dot{m} = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \rho \lim_{\Delta t \to 0} \frac{\Delta \forall}{\Delta t} = \rho Q = \rho \overline{V} A$$

$$\dot{m} = \int_A \rho \vec{V}.d\vec{A}$$

## EXAMPLE 5.2 FLOW IN SLOPING CHANNEL

Water flows in a channel that has a slope of 30°. If the velocity is assumed to be constant, 12 m/s, and if a depth of 60 cm is measured along a vertical line, what is the discharge per meter of width of the channel?

**Sketch:**




*Problem Definition*

**Situation:** Channel slope of 30°. Velocity is 12 m/s and vertical depth is 60 cm.

**Find:** Discharge per meter width ($m^2/s$).

**Assumptions:** Velocity is uniformly distributed across channel.

### *Plan*

Use Eq. (5.7) with area based on 1 meter width.

### *Solution*

$$\begin{aligned} Q = \mathbf{V} \cdot \mathbf{A} &= V \cos 30 \times A \\ &= 12 \text{ m/s} \times \cos 30^\circ \times 0.6 \text{ m} \\ &= \boxed{6.24 \text{ m}^3\text{/s per meter}} \end{aligned}$$

### *Review*

The discharge per unit width is usually designated as $q$.

## EXAMPLE 5.3 DISCHARGE IN CHANNEL WITH NON-UNIFORM VELOCITY DISTRIBUTION

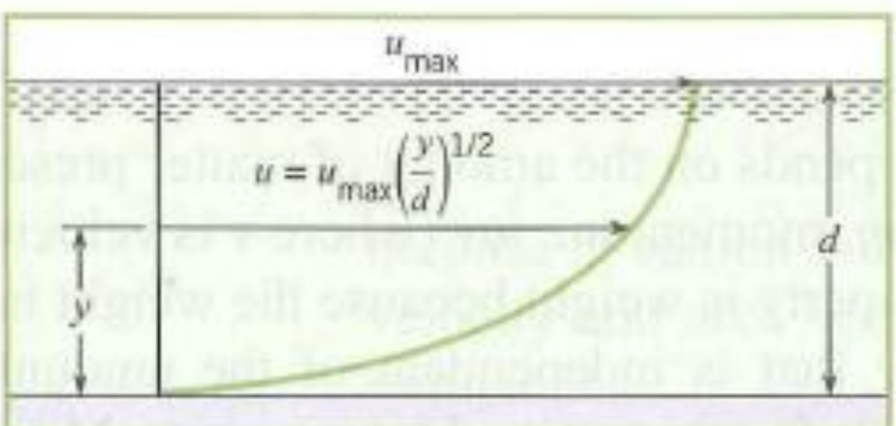
The water velocity in the channel shown in the accompanying figure has a distribution across the vertical section equal to $u/u_{max} = (y/d)^{1/2}$. What is the discharge in the channel if the water is 2 m deep ($d = 2$ m), the channel is 5 m wide, and the maximum velocity is 3 m/s?

### *Problem Definition*

**Situation:** Water flows in a 2 m by 5 m channel with a given velocity distribution.

**Find:** Discharge (in $m^3/s$).

**Sketch:**



*Solution*

Discharge equation

$$Q = \int_0^d u\, dA$$

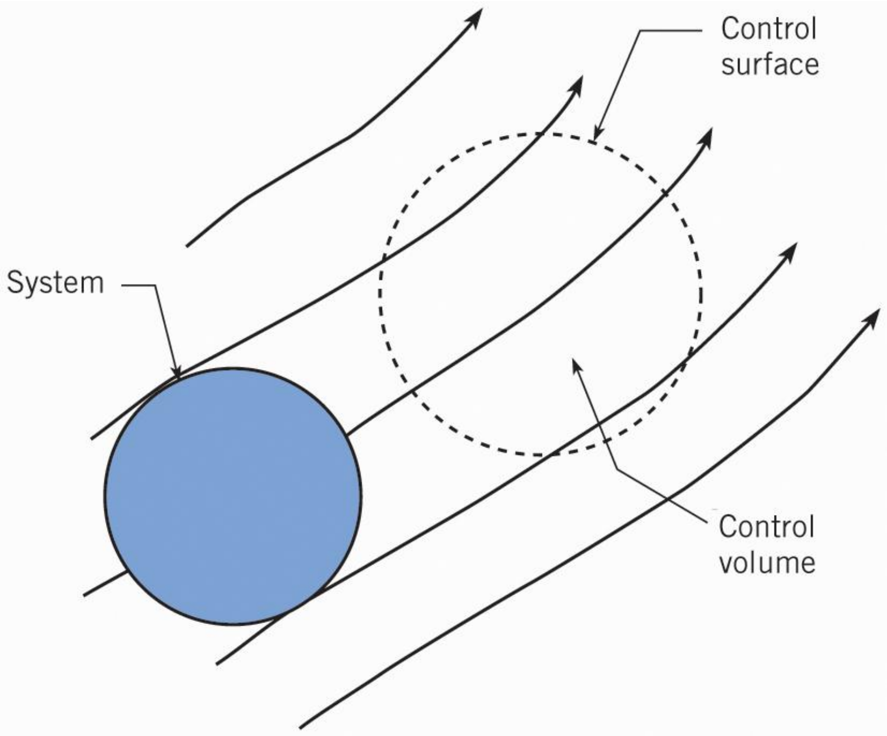
Channel is 5 m wide, so differential area is $dA = 5dy$. Using given velocity distribution,

$$\begin{aligned} Q &= \int_0^2 u_{\max}(y/d)^{1/2} 5\, dy \\ &= \frac{5u_{\max}}{d^{1/2}} \int_0^2 y^{1/2}\, dy \\ &= \frac{5u_{\max}}{d^{1/2}} \frac{2}{3} y^{3/2} \Big|_0^2 \\ &= \frac{5 \times 3}{2^{1/2}} \times \frac{2}{3} \times 2^{3/2} = \boxed{20\ \mathrm{m^3/s}} \end{aligned}$$

# 5.2 Control volume approach

System and control volume

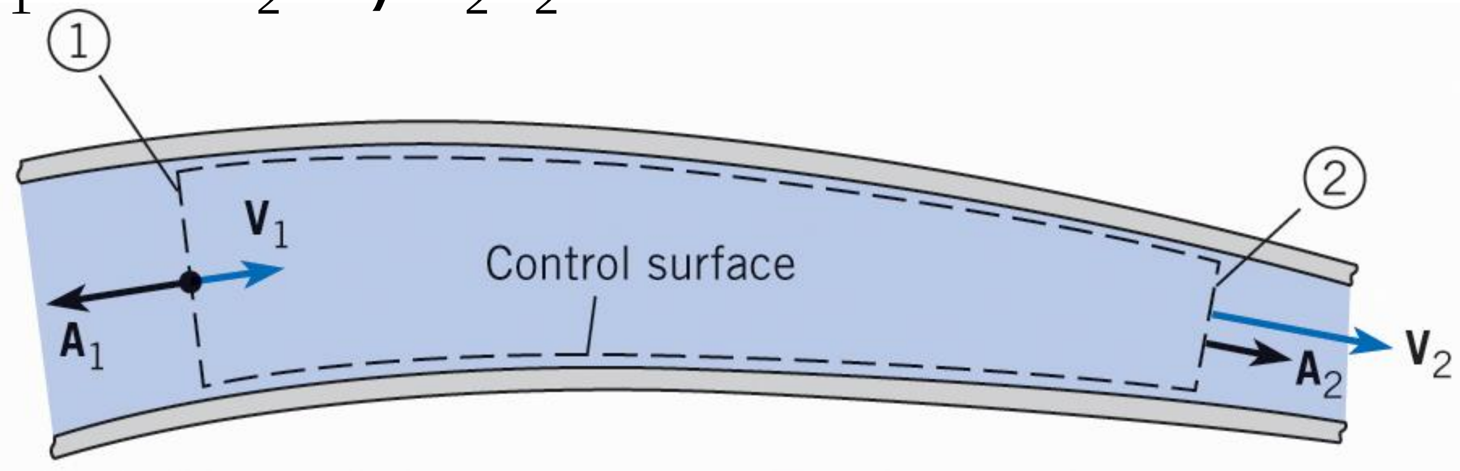
- **Intensive Property:**
  The property of a fluid that does not depend on the mass.
- **Extensive Property:**
  The property of a fluid that depends on the mass.




$$B_{cv} = \int_{cv} b\,dm = \int_{cv} b\rho\, d\forall$$

# Property transport across the control surface

$$\dot{m}_1 = \rho A_1 V_1 \qquad \dot{m}_2 = \rho A_2 V_2$$



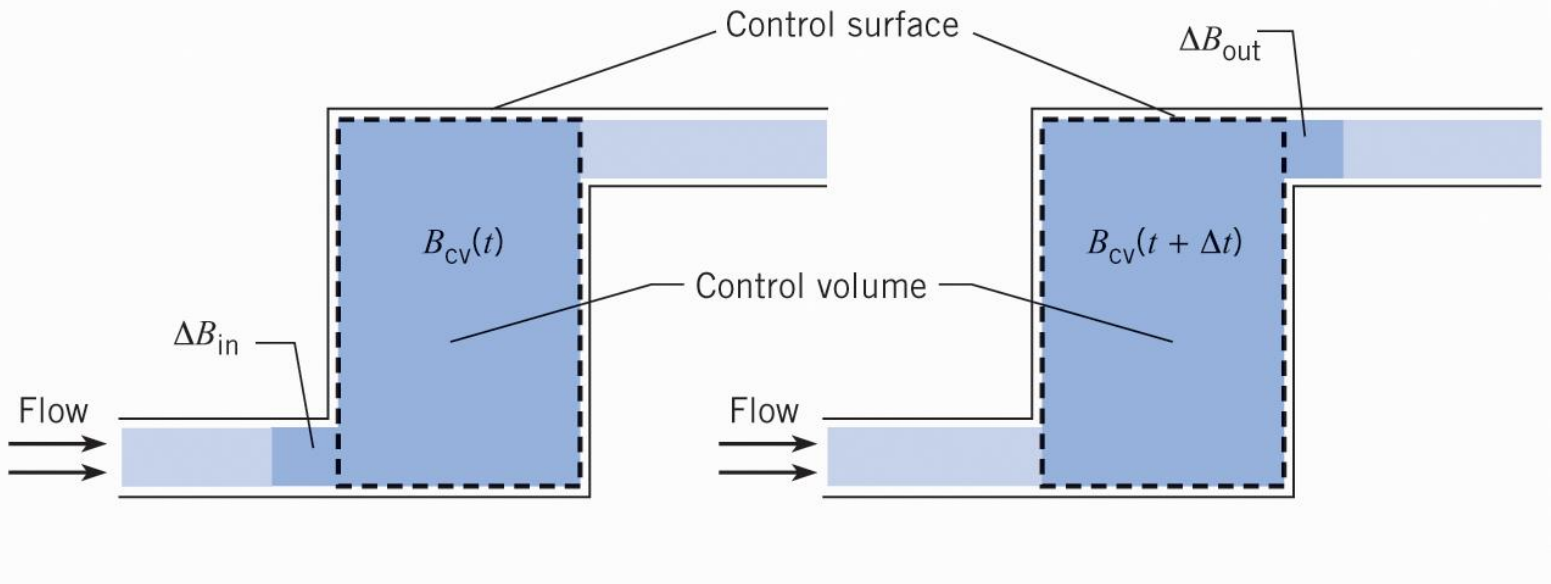

Net mass outflow rate= $\dot{m}_2 - \dot{m}_1 = \rho_2 V_2 A_2 - \rho_1 V_1 A_1$

$$= \rho_2 \vec{V}_2 . \vec{A}_2 + \rho_1 \vec{V}_1 . \vec{A}_1$$

$$= \sum_{cs} \rho \vec{V} . \vec{A}$$

Net mass outflow rate of the extensive property= $\sum_{cs} b \rho \vec{V} . \vec{A}$

# Reynolds transport theorem



System at time $t$ System at time $t+\Delta t$

$$B_{sys}(t) = B_{cv}(t) + \Delta B_{in}$$

$$B_{sys}(t+\Delta t) = B_{cv}(t+\Delta t) + \Delta B_{out}$$

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \to 0}\left[\frac{B_{sys}(t+\Delta t) - B_{sys}(t)}{\Delta t}\right]$$

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \to 0}\left[\frac{B_{cv}(t+\Delta t) - B_{cv}(t) + \Delta B_{out} - \Delta B_{in}}{\Delta t}\right]$$

$$\frac{dB_{sys}}{dt} = \frac{dB_{cv}}{dt} + \dot{B}_{net}$$

$$\underbrace{\frac{dB_{sys}}{dt}}_{Lagrangian} = \underbrace{\frac{d}{dt}\int_{cv} b\rho d\forall + \int_{cs} b\rho \vec{V}.d\vec{A}}_{Eulerian}$$

# 5.3 Continuity equation

$$B = m, \quad b = 1$$

$$\frac{dm_{sys}}{dt} = \frac{d}{dt}\int_{cv} \rho d\forall + \int_{cs} \rho \vec{V}.d\vec{A}$$

$$\frac{dm_{sys}}{dt} = 0 \quad \text{(law of conservation of mass)}$$

$$\frac{d}{dt}\int_{cv} \rho d\forall + \int_{cs} \rho \vec{V}.d\vec{A} = 0$$

## EXAMPLE 5.4 MASS ACCUMULATION IN A TANK

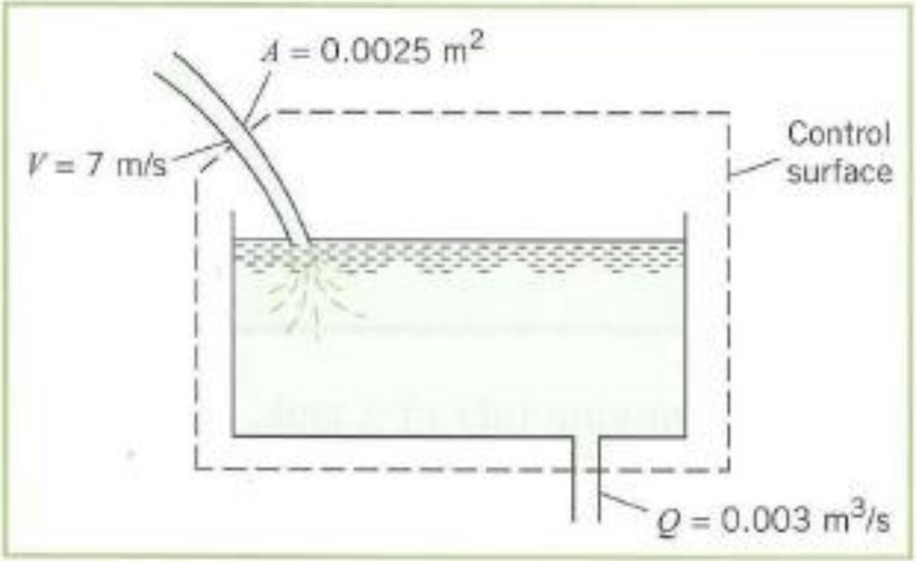
A jet of water discharges into an open tank, and water leaves the tank through an orifice in the bottom at a rate of 0.003 $m^3/s$. If the cross-sectional area of the jet is 0.0025 $m^2$ where the velocity of water is 7 m/s, at what rate is water accumulating in (or evacuating from) the tank?

*Problem Definition*

**Situation:** Jet of water (7 m/s at 0.0025 $m^2$) entering tank and water leaving at 0.003 $m^3/s$ through orifice.

**Find:** Rate of accumulation (or evacuation) in tank (kg/s).

**Sketch:**




**Assumptions:** Water density is 1000 $kg/m^3$.

Solution

1. Continuity equation

$$\frac{d}{dt}m_{cv} + \sum_{cs}\dot{m}_o - \sum_{cs}\dot{m}_i = 0$$

Because there is only one inlet and outlet, the equation reduces to

$$\frac{d}{dt}m_{cv} = \dot{m}_i - \dot{m}_o$$

2. Term-by-term analysis
   - The inlet mass flow rate is calculated using Eq. (5.5)

$$\begin{aligned}\dot{m}_i &= \rho VA \\ &= 1000\ \text{kg/m}^3 \times 7\ \text{m/s} \times 0.0025\ \text{m}^2 \\ &= 17.5\ \text{kg/s}\end{aligned}$$

   - Outlet flow rate is

$$\dot{m}_o = \rho Q = 1000\ \text{kg/m}^3 \times 0.003\ \text{m}^3/\text{s} = 3\ \text{kg/s}$$

3. Accumulation rate:

$$\begin{aligned}\frac{dm_{cv}}{dt} &= 17.5\ \text{kg/s} - 3\ \text{kg/s} \\ &= \boxed{14.5\ \text{kg/s}}\end{aligned}$$

## EXAMPLE 5.6 WATER LEVEL DROP RATE IN DRAINING TANK

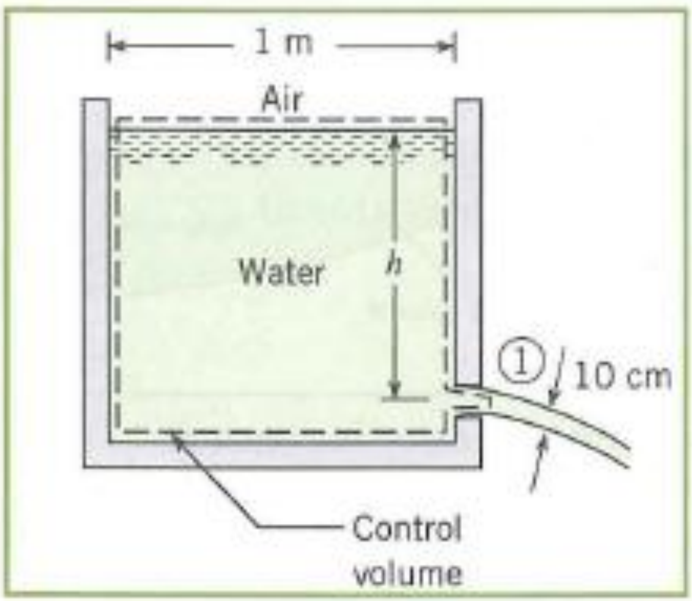
A 10 cm jet of water issues from a 1 m diameter tank. Assume that the velocity in the jet is $\sqrt{2gh}$ m/s where h is the elevation of the water surface above the outlet jet. How long will it take for the water surface in the tank to drop from $h_0 = 2$ m to $h_f = 0.50$ m?

*Problem Definition*

**Situation:** Water draining by a 10 cm jet from 1 m diameter tank.

**Find:** Time (in seconds) to drain from depth of 2 m to 0.5 m.

**Sketch:**



*Solution*

1. Continuity equation

$$\frac{d}{dt}m_{cv} + \sum_{cs} \dot{m}_o - \sum_{cs} \dot{m}_i = 0$$

2. Term-by-term analysis
   - Accumulation rate term

$$dm_{cv} = \rho A_T dh$$

$$\frac{dm_{cv}}{dt} = \rho A_T \frac{dh}{dt}$$

where $A_T$ is cross-sectional area of tank.

- Inlet mass flow rate with no inflow is

$$\sum_{cs} \dot{m}_i = 0$$

- Outlet mass flow rate

$$\sum_{cs} \dot{m}_o = \rho A_1 V_1 = \rho \sqrt{2gh} A_1$$

Substitution of terms in continuity equation:

$$-\rho V_1 A_1 = \frac{d(\rho A_T h)}{dt}$$

$$-\sqrt{2gh} A_1 = A_T \frac{dh}{dt}$$

3. Equation for elapsed time:

- Separating variables

$$dt = \frac{-A_T}{\sqrt{2g} A_1} \frac{dh}{\sqrt{h}} \quad \text{or} \quad dt = \frac{-A_T}{\sqrt{2g} A_1} h^{-1/2} dh$$

- Integrating

$$t = \frac{-2A_T}{\sqrt{2g}A_1}h^{1/2} + C$$

- Substituting in initial condition, $h(0) = h_0$, and final condition, $h(t) = h_f$, and solving for time

$$t = \frac{2A_T}{\sqrt{2g}A_1}(h_0^{1/2} - h_f^{1/2})$$

4. Time calculation:

- Evaluating tank and outlet areas

$$A_1 = \frac{\pi}{4}(0.10\text{ m})^2 = 0.01\left(\frac{\pi}{4}\right)\text{m}^2$$

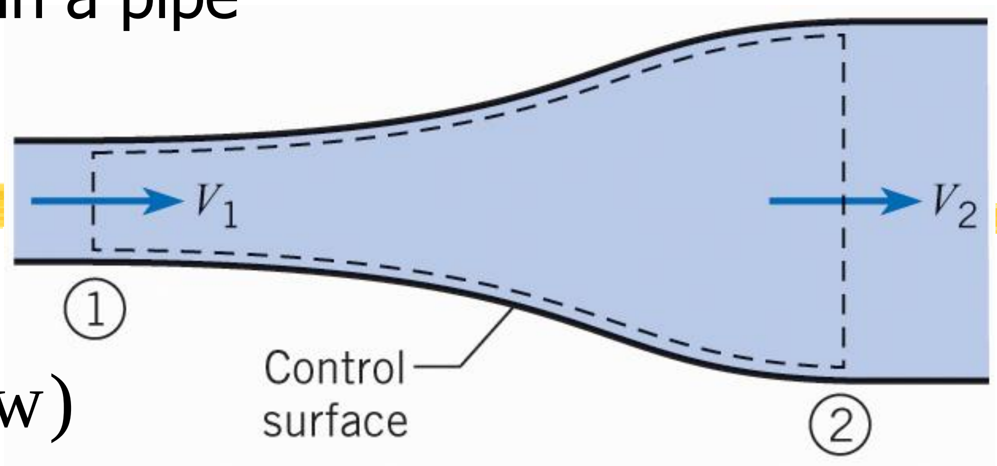
$$A_T = \frac{\pi}{4}D^2 = \frac{\pi}{4}\times(1\text{ m})^2 = \frac{\pi}{4}\text{ m}^2$$

- Elapsed time

$$t = \frac{2(\pi/4)\text{ m}^2}{\sqrt{2\times 9.81\text{ m/s}^2}(\pi/4\times 0.01\text{ m}^2)}(\sqrt{2\text{ m}} - \sqrt{0.5\text{ m}})$$

$$= \boxed{31.9\text{ s}}$$

# Continuity equation for flow in a pipe




$$\frac{d}{dt}\int_{cv}\rho d\forall + \int_{cs}\rho\vec{V}.d\vec{A} = 0$$

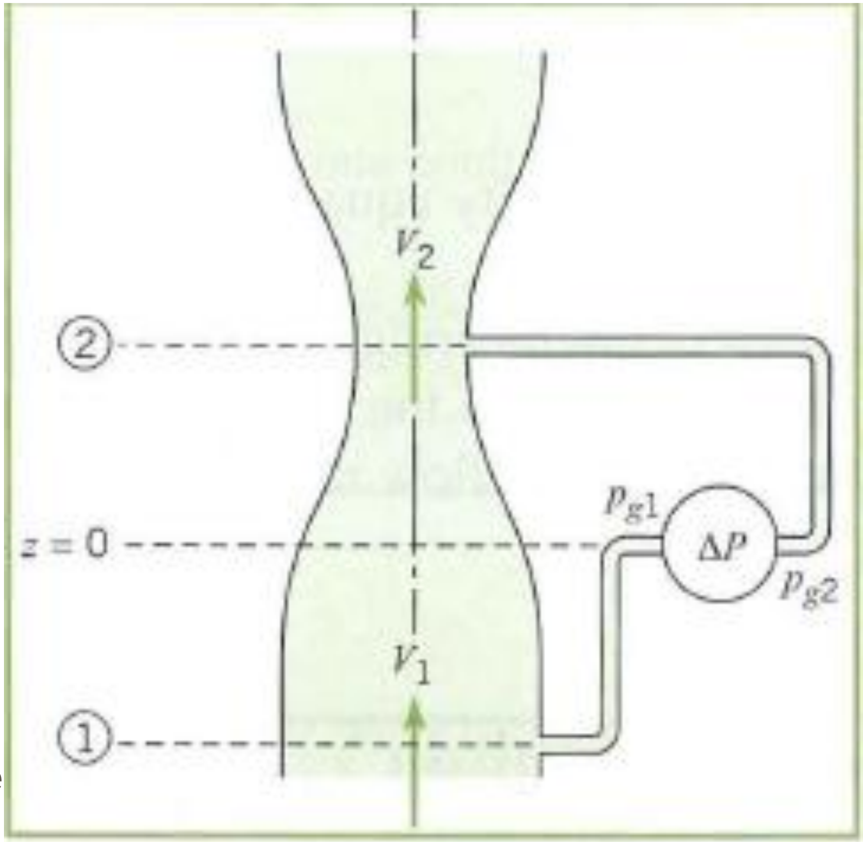
$$\frac{d}{dt}\int_{cv}\rho d\forall = 0 \quad \text{(steady flow)}$$

$$\int_{cs}\rho\vec{V}.d\vec{A} = 0$$

$$\sum_{cs}\dot{m}_{out} = \sum_{cs}\dot{m}_{in}$$

# EXAMPLE 5.9 WATER FLOW THROUGH A VENTURIMETER .

Water with a density of 1000 $kg/m^3$ flows through a vertical venturimeter as shown. A pressure gage is connected across two taps in the pipe (station 1) and the throat (station 2). The area ratio $A_{throat}/A_{pipe}$ is 0.5. The velocity in the pipe is 10 m/s. Find the pressure difference recorded by the pressure gage. Assume the flow has a uniform velocity distribution and that viscous effects are not important.



*Solution*

1. The Bernoulli equation

$$p_1 + \gamma z_1 + \rho\frac{V_1^2}{2} = p_2 + \gamma z_2 + \rho\frac{V_2^2}{2}$$

Rewrite the equation in terms of piezometric pressure.

$$p_{z_1} - p_{z_2} = \frac{\rho}{2}(V_2^2 - V_1^2)$$

$$= \frac{\rho V_1^2}{2}\left(\frac{V_2^2}{V_1^2} - 1\right)$$

2. Continuity equation $V_2/V_1 = A_1/A_2$

$$p_{z_1} - p_{z_2} = \frac{\rho V_1^2}{2}\left(\frac{A_1^2}{A_2^2} - 1\right)$$

$$= \frac{1000\ \mathrm{kg/m^3}}{2} \times (10\ \mathrm{m/s})^2 \times (2^2 - 1)$$

$$= 150\ \mathrm{kPa}$$

3. Gage is located at zero elevation. Apply hydrostatic equation through static fluid in gage line between gage attachment point where the pressure is $p_{g_1}$ and station 1 where the gage line is tapped into the pipe,

$$p_{z_1} = p_{g_1}$$

Also $p_{z_2} = p_{g_2}$ so

$$\Delta p_{\text{gage}} = p_{g_1} - p_{g_2} = p_{z_1} - p_{z_2} = \boxed{150\ \text{kPa}}$$