Chapter 3
Fluid Statics

Ahmad Sana
Department of Civil and Architectural Engineering
Sultan Qaboos University
Sultanate of Oman
Email: sana@squ.edu.om
Webpage: http://ahmadsana.tripod.com

Significant learning outcomes

Conceptual Knowledge

- Describe pressure and pressure distribution.
- Describe gage, absolute, and vacuum pressure.
- List the steps used to derive the hydrostatic differential equation.

Procedural Knowledge

- Apply the hydrostatic equation and the manometer equations to predict pressure.
- Apply the panel equations to predict forces and moments.
- Apply the buoyancy equation to predict forces.

Applications (typical)

- For applications involving the atmosphere, the ocean, manometers, and hydraulic machines, find pressure values and distributions.
- For structures and components subjected to hydrostatic loading, find forces and moments.

3.1 Pressure

- A normal force exerted by a fluid per unit area is called fluid pressure.

$$p = \lim_{\Delta A \to 0} \frac{\left|\Delta \vec{F}_{normal}\right|}{\Delta A} = \frac{dF_{normal}}{dA}$$

- Units

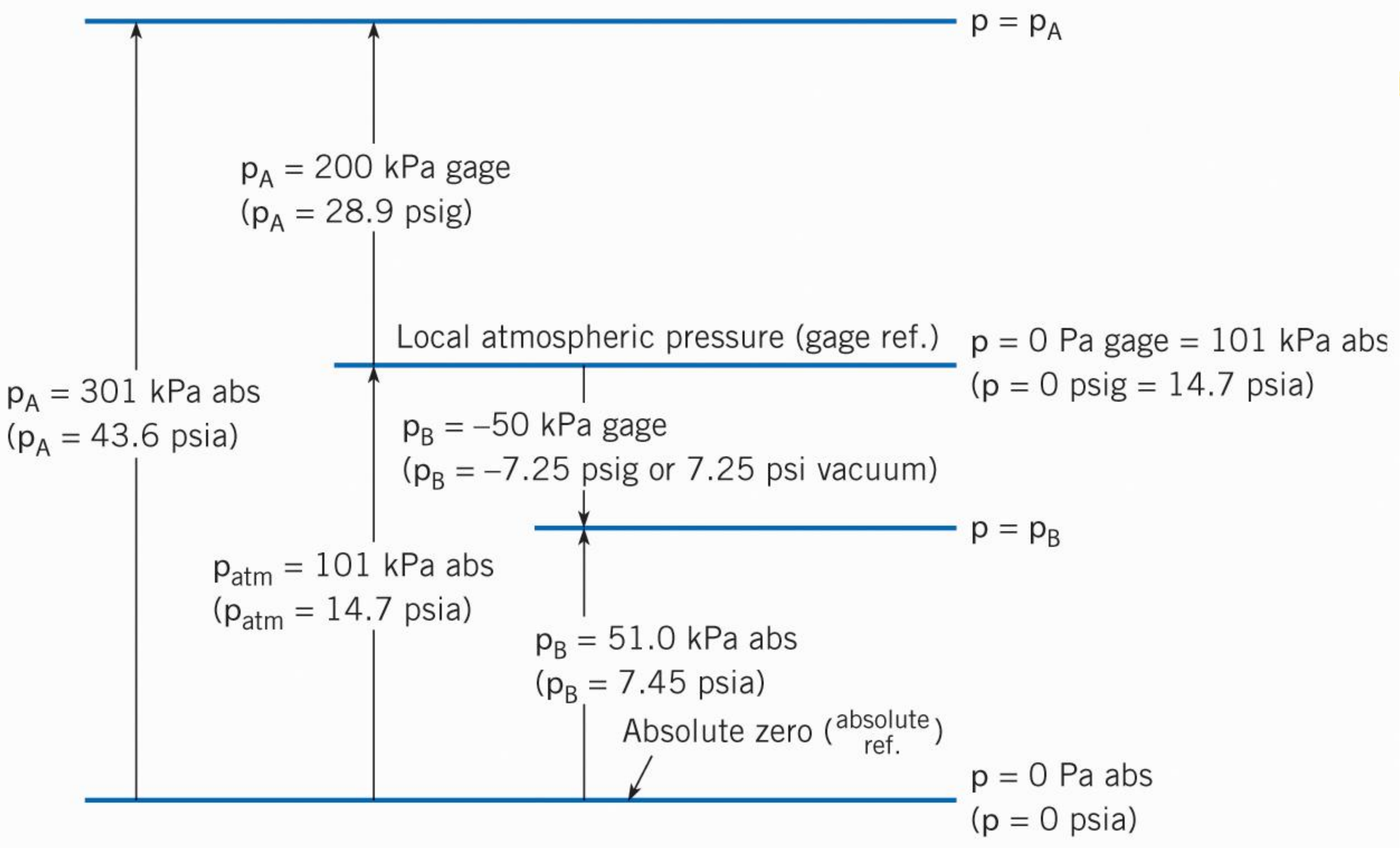
1 Pa = 1 N/m^2

1 bar = 10^5 Pa = 100 kPa

1 atm= 101.325 kPa= 1.01325 bar

1 kgf/cm^2= 9.807 N/cm^2=98.07 kPa=0.9807 bar

Absolute pressure, gage pressure and vacuum pressure



Hydraulic machines

Pascal's law: Pressure applied to an enclosed and continuous body of fluid is transmitted undiminished to every portion of that fluid and to the walls of the containing vessel.

EXAMPLE 3.1 LOAD LIFTED BY A HYDRAULIC JACK

A hydraulic jack has the dimensions shown. If one exerts a force F of 100 N on the handle of the jack, what load, F_2, can the jack support? Neglect lifter weight.

Problem Definition

Situation: A force of $F = 100$ N is applied to the handle of a jack.

Find: Load F_2 in kN that the jack can lift.

Assumptions: Weight of the lifter component (see sketch) is negligible.

Solution

1. Moment equilibrium

$$\sum M_C = 0$$

$$(0.33\text{ m}) \times (100\text{ N}) - (0.03\text{ m})F_1 = 0$$

$$F_1 = \frac{0.33\text{ m} \times 100\text{ N}}{0.03\text{ m}} = 1100\text{ N}$$

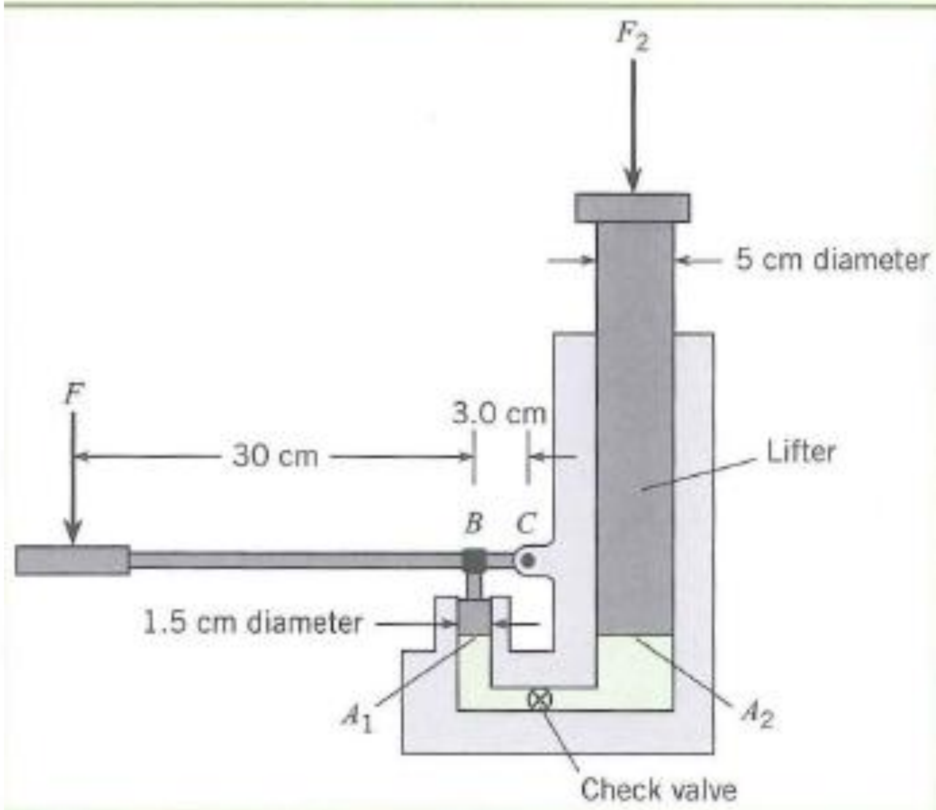
2. Force equilibrium (small piston)

$$\sum F_{\text{small piston}} = p_1 A_1 - F_1 = 0$$

$$p_1 A_1 = F_1 = 1100\text{ N}$$

Thus

$$p_1 = \frac{F_1}{A_1} = \frac{1100\text{ N}}{\pi d^2/4} = 6.22 \times 10^6\text{ N/m}^2$$



3. Force equilibrium (lifter)

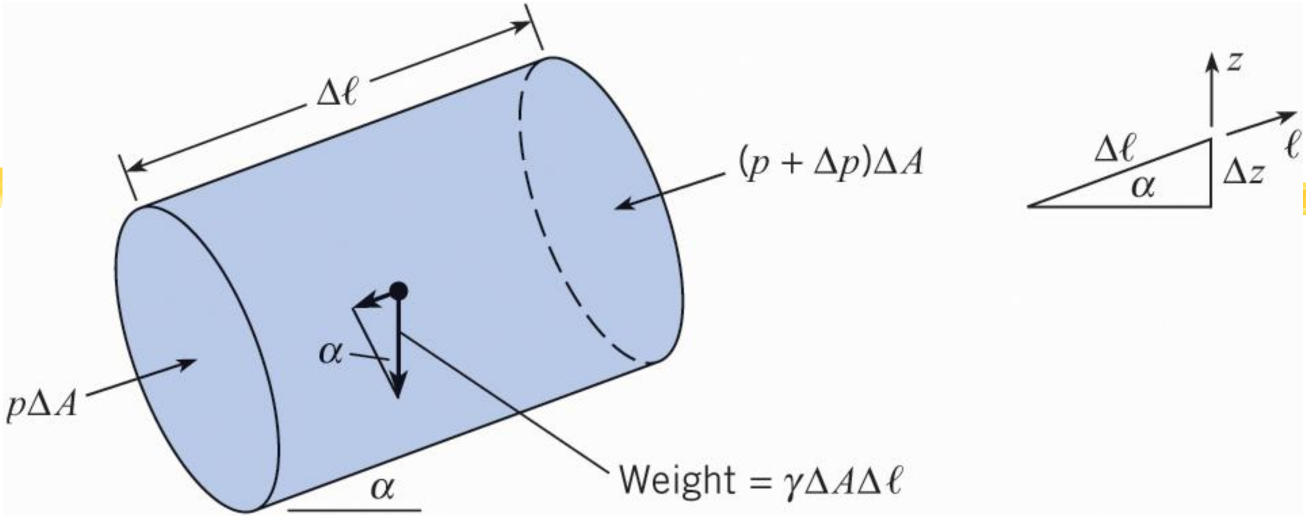
- Note that $p_1 = p_2$ because they are at the same elevation (this fact will be established in the next section).
- Apply force equilibrium:

$$\sum F_{\text{lifter}} = F_2 - p_1 A_2 = 0$$

$$F_2 = p_1 A_2 = \left(6.22 \times 10^6 \ \frac{\text{N}}{\text{m}^2}\right)\left(\frac{\pi}{4} \times (0.05 \text{ m})^2\right) = \boxed{12.2 \text{ kN}}$$

3.2 Pressure variation with elevation

Hydrostatic differential equation



$$\sum F_l = 0 \quad \Rightarrow \quad F_{pressure} - F_{weight} = 0$$

$$p\Delta A - (p + \Delta p)\Delta A - \gamma \Delta A \Delta l \sin\alpha = 0$$

$$\frac{\Delta p}{\Delta l} = -\gamma \sin\alpha \quad \Rightarrow \quad \frac{\Delta p}{\Delta z} = -\gamma \quad \left(\because \ \sin\alpha = \frac{\Delta z}{\Delta l} \right)$$

$$\lim_{\Delta z \to 0} \frac{\Delta p}{\Delta z} = -\gamma \quad \Rightarrow \quad \frac{dp}{dz} = -\gamma$$

Hydrostatic equation

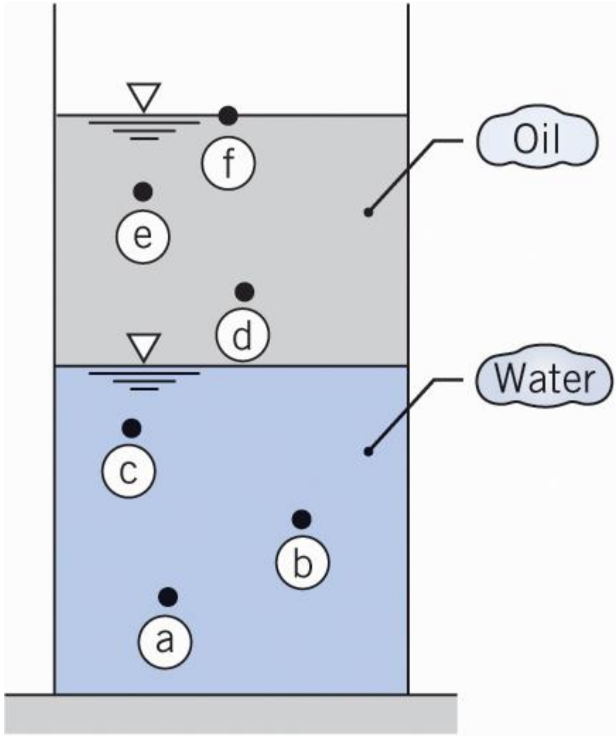
$$\frac{dp}{dz} = -\gamma \quad \text{integrating}$$

$$p = -\gamma z + \text{constant}$$

$$p + \gamma z = \text{constant} = p_z = \text{piezometric pressure}$$

$$p_1 + \gamma z_1 = p_2 + \gamma z_2$$

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2 = h = \text{piezometric head}$$



$$h_a = h_b = h_c$$

$$h_d = h_e = h_f$$

$$h_c \neq h_d$$

EXAMPLE 3.3 PRESSURE IN TANK WITH TWO FLUIDS

Oil with a specific gravity of 0.80 forms a layer 0.90 m deep in an open tank that is otherwise filled with water. The total depth of water and oil is 3 m. What is the gage pressure at the bottom of the tank?

Problem Definition

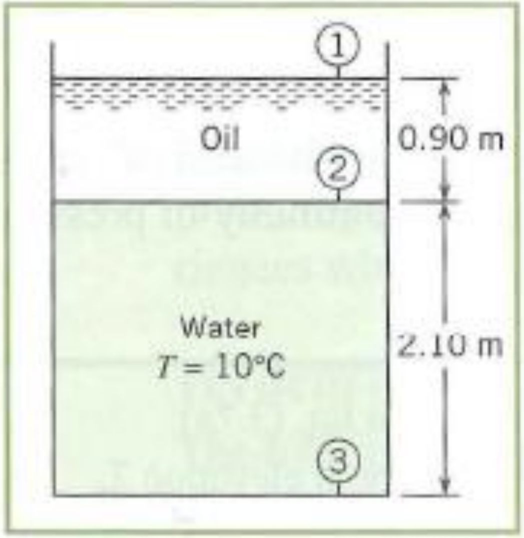
Situation: Oil and water are contained in a tank.

Find: Pressure (kPa gage) at the bottom of the tank.

Properties:

1. Oil (10°C), $S = 0.8$.
2. Water (10°C), Table A.5: $\gamma = 9810\ \mathrm{N/m^3}$.

Sketch:



Solution

1. Hydrostatic equation (oil)

$$\frac{p_1}{\gamma_{oil}} + z_1 = \frac{p_2}{\gamma_{oil}} + z_2$$

$$\frac{0 \text{ Pa}}{\gamma_{oil}} + 3 \text{ m} = \frac{p_2}{0.8 \times 9810 \text{ N/m}^3} + 2.1 \text{ m}$$

$$p_2 = 7.063 \text{ kPa}$$

2. Oil-water interface

$$p_2\big|_{oil} = p_2\big|_{water} = 7.063 \text{ kPa}$$

3. Hydrostatic equation (water)

$$\frac{p_2}{\gamma_{water}} + z_2 = \frac{p_3}{\gamma_{water}} + z_3$$

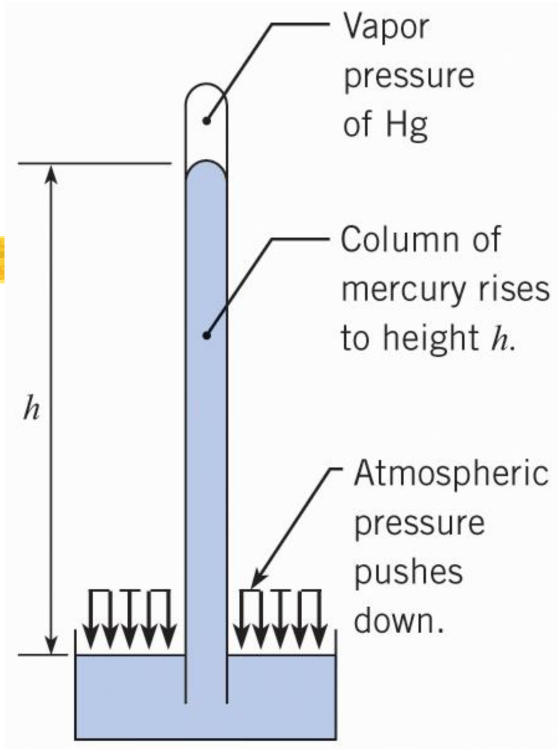
$$\frac{7.063 \times 10^3 \text{ Pa}}{9810 \text{ N/m}^3} + 2.1 \text{ m} = \frac{p_3}{9810 \text{ N/m}^3} + 0 \text{ m}$$

$$\boxed{p_3 = 27.7 \text{ kPa gage}}$$

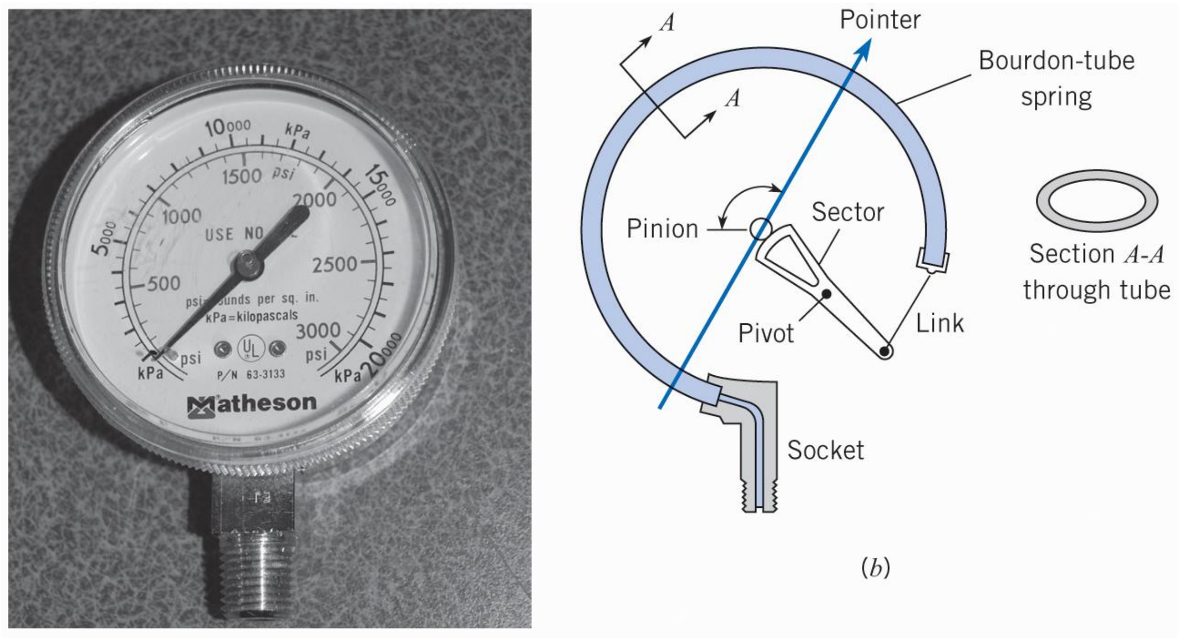
3.3 Pressure measurements

Barometer: an instrument to measure atmospheric pressure

$$p_{atm} = \gamma_{Hg} h + p_{v,Hg} \approx \gamma_{Hg} h$$

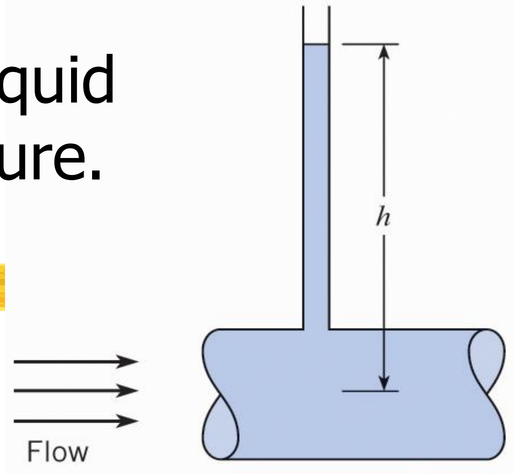


Bourdon-Tube gage: an instrument to measure pressure by sensing the deflection of a coiled tube.



(a) (b)

Piezometer: a vertical tube in which a liquid rises in response to a positive gage pressure.



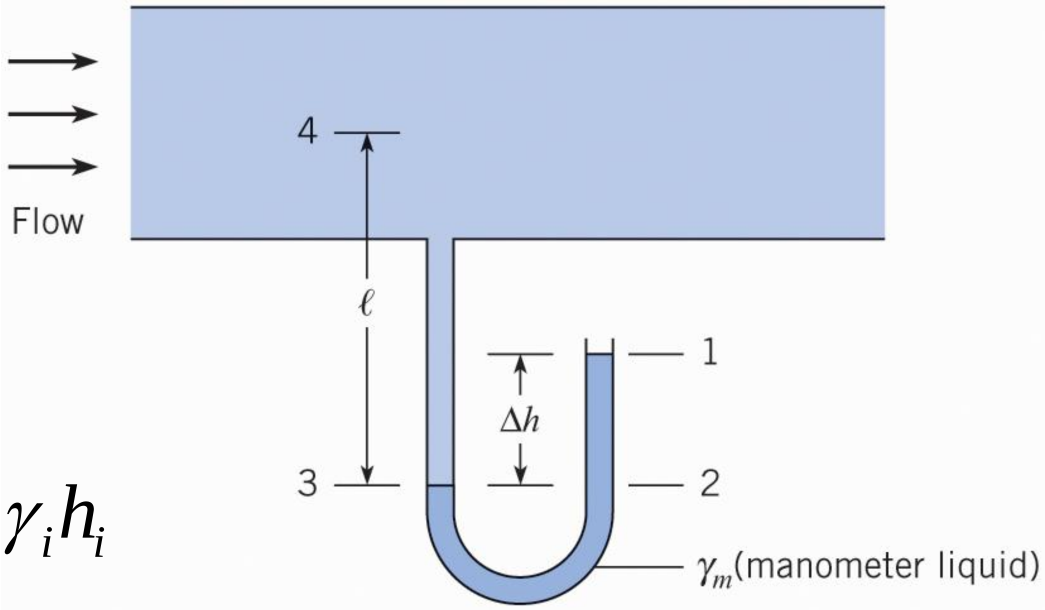
Manometer: a U shaped tube to measure pressure by the column of a liquid.

$$p_3 = p_2$$

$$p_3 = p_4 + \gamma l$$

$$p_2 = \gamma_m \Delta h + 0$$

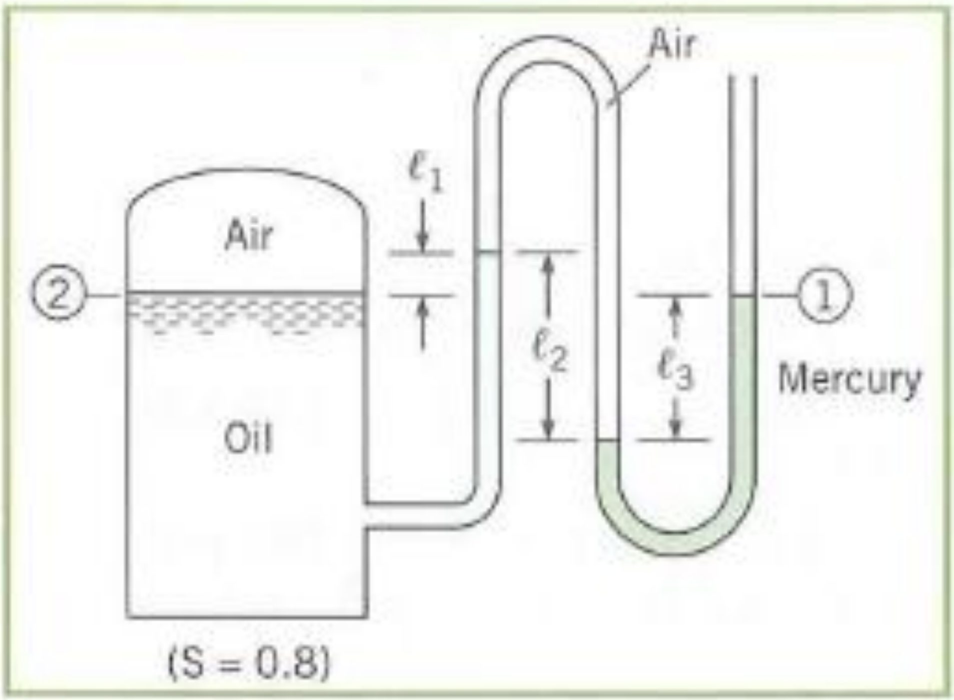
$$p_4 = \gamma_m \Delta h - \gamma l$$



$$p_2 = p_1 + \sum_{down} \gamma_i h_i - \sum_{up} \gamma_i h_i$$

EXAMPLE 3.7 MANOMETER ANALYSIS

Sketch: What is the pressure of the air in the tank if $\ell_1 = 40$ cm, $\ell_2 = 100$ cm, and $\ell_3 = 80$ cm?



Problem Definition

Situation: A tank is pressurized with air.

Find: Pressure (kPa gage) in the air.

Assumptions: Neglect the pressure change in the air column.

Properties:

1. Oil:

$$\gamma_{oil} = S\gamma_{water} = 0.8 \times 9810\ \text{N/m}^3 = 7850\ \text{N/m}^3.$$

2. Mercury, Table A.4: $\gamma = 133{,}000\ \text{N/m}^3$.

Plan

Apply the manometer equation (3.18) from elevation 1 to elevation 2.

Solution

Manometer equation

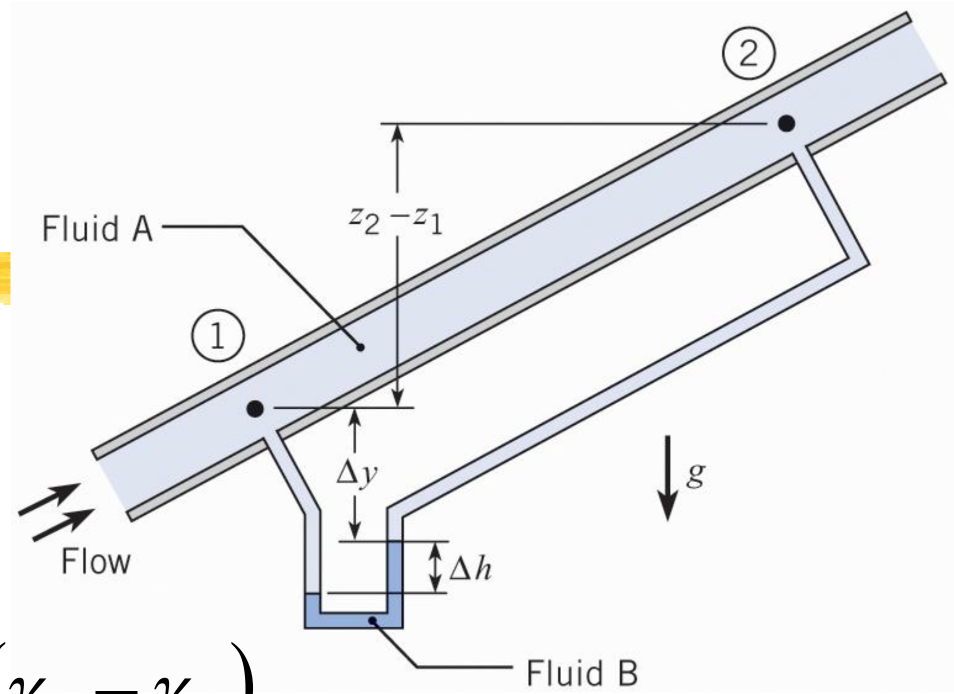
$$p_1 + \sum_{\text{down}} \gamma_i h_i - \sum_{\text{up}} \gamma_i h_i = p_2$$

$$p_1 + \gamma_{mercury}\ell_3 - \gamma_{air}\ell_2 + \gamma_{oil}\ell_1 = p_2$$

$$0 + (133{,}000\ \text{N/m}^3)(0.8\ \text{m}) - 0 + (7850\ \text{N/m}^3)(0.4\ \text{m}) = p_2$$

$$\boxed{p_2 = p_{air} = 110\ \text{kPa gage}}$$

$$p_2 = p_1 + \sum_{down} \gamma_i h_i - \sum_{up} \gamma_i h_i$$



$$p_2 = p_1 + \gamma_A(\Delta y + \Delta h)$$
$$- \gamma_B \Delta h - \gamma_A(\Delta y + z_2 - z_1)$$

$$(p_1 + \gamma_A z_1) - (p_2 + \gamma_A z_2) = \Delta h(\gamma_B - \gamma_A)$$

$$\left(\frac{p_1}{\gamma_A} + z_1\right) - \left(\frac{p_2}{\gamma_A} + z_2\right) = \Delta h\left(\frac{\gamma_B}{\gamma_A} - 1\right)$$

$$h_1 - h_2 = \Delta h\left(\frac{\gamma_B}{\gamma_A} - 1\right)$$

EXAMPLE 3.8 CHANGE IN PIEZOMETRIC HEAD FOR PIPE FLOW

A differential mercury manometer is connected to two pressure taps in an inclined pipe as shown in Fig. 3.12. Water at 280 K is flowing through the pipe. The deflection of mercury in the manometer is 2.5 cm. Find the change in piezometric pressure and piezometric head between points 1 and 2.

Problem Definition

Situation: Water is flowing in a pipe.

Find:

1. Change in piezometric head (m) between points 1 and 2.
2. Change in piezometric pressure (kPa gage) between 1 and 2.

Properties:

1. Water (10°C), Table A.5, $\gamma_{water} = 9810\ N/m^3$.
2. Mercury, Table A.4, $\gamma_{Hg} = 133{,}000\ N/m^3$.

Plan

1. Find difference in the piezometric head using Eq. (3.19).
2. Relate piezometric head to piezometric pressure using Eq. (3.10).

Solution

Difference in piezeometric head

$$h_1 - h_2 = \Delta h\left(\frac{\gamma_{Hg}}{\gamma_{water}} - 1\right) = (0.025 \text{ m})\left(\frac{133{,}000 \text{ N/m}^3}{9810 \text{ N/m}^3} - 1\right)$$

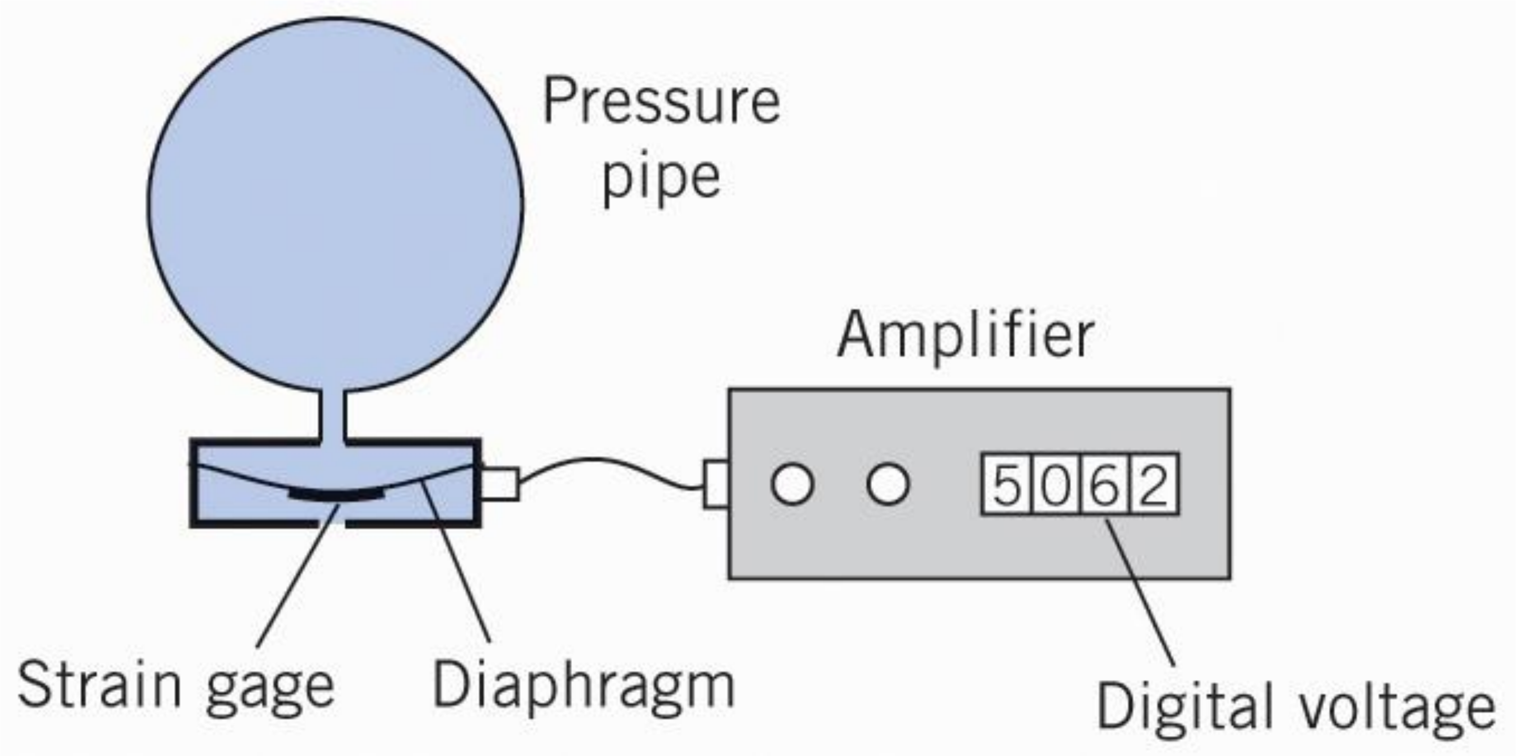
$$= \boxed{0.31 \text{ m}}$$

Piezometric pressure

$$p_z = h\gamma_{water}$$

$$= (0.31 \text{ m})(9810 \text{ N/m}^3) = \boxed{3.04 \text{ kPa}}$$

Pressure transducer: a device that converts pressure to an electrical signal.



3.4 Forces on Plane Surfaces

Magnitude of resultant hydrostatic force

$$p = \gamma y \sin\alpha$$

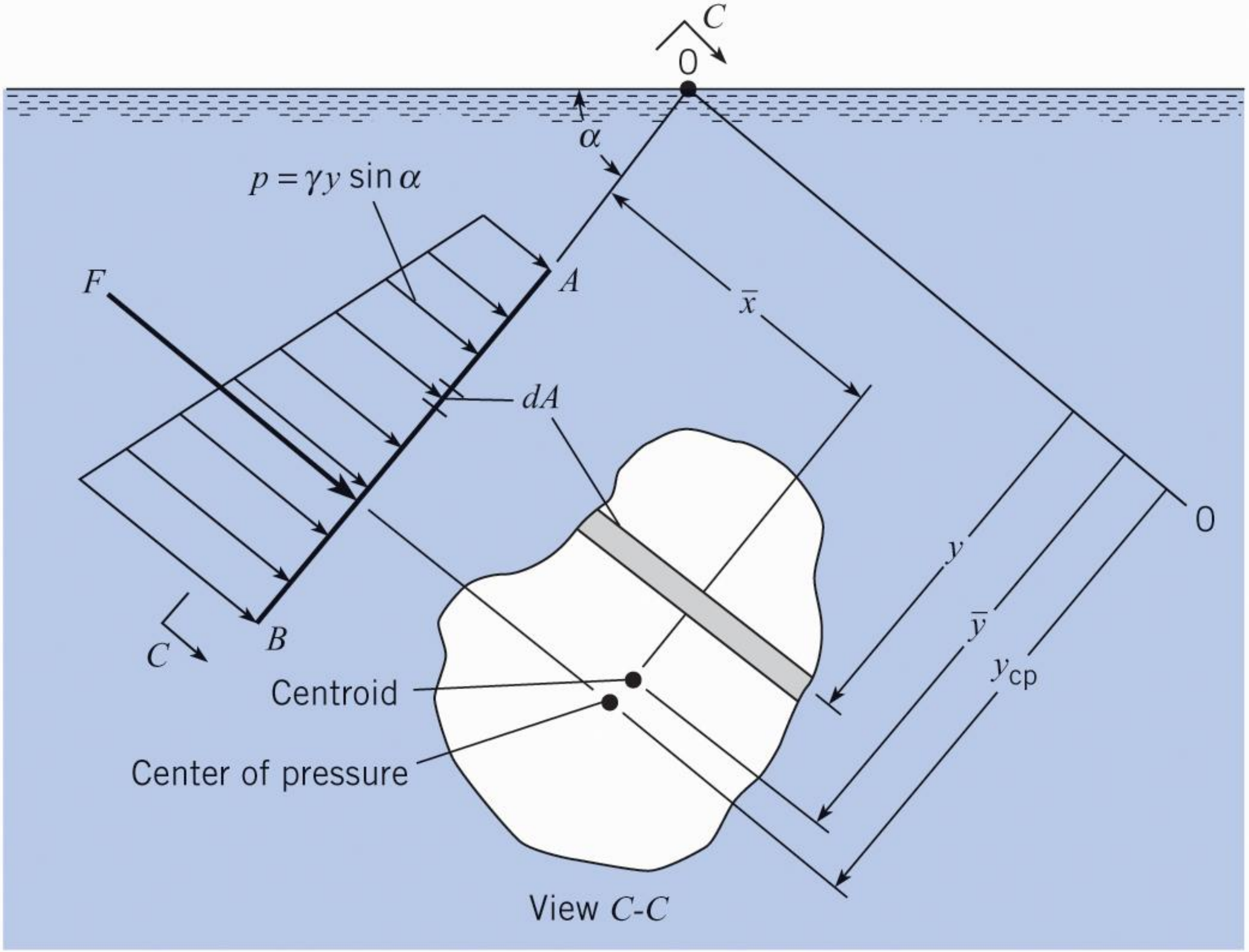
$$dF = p\,dA$$

$$F = \int_A p\,dA$$

$$= \int_A (\gamma\, y \sin\alpha)\, dA$$

$$= \gamma \sin\alpha \int_A y\,dA$$

$$F = (\gamma \sin\alpha)\bar{y}A = (\gamma \sin\alpha\bar{y})A = \bar{p}A$$



The magnitude of the resultant force acting on a completely submerged plane surface in a homogeneous fluid is equal to the product of the pressure at the centroid of the surface and the area of the surface.

Line of action of resultant force

$$y_{cp}F = \int_A yp\, dA = \gamma \sin\alpha \int_A y^2 dA = \gamma \sin\alpha\, I_0$$

$$I_0 = \bar{I} + \bar{y}^2 A$$

$$y_{cp}F = \gamma \sin\alpha \left(\bar{I} + \bar{y}^2 A\right)$$

$$y_{cp}\left(\gamma \bar{y} \sin\alpha\right) = \gamma \sin\alpha \left(\bar{I} + \bar{y}^2 A\right)$$

$$y_{cp} = \bar{y} + \frac{\bar{I}}{\bar{y}A}$$

EXAMPLE 3.10 FORCE TO OPEN AN ELLIPTICAL GATE

An elliptical gate covers the end of a pipe 4 m in diameter. If the gate is hinged at the top, what normal force F is required to open the gate when water is 8 m deep above the top of the pipe and the pipe is open to the atmosphere on the other side? Neglect the weight of the gate.

Problem Definition

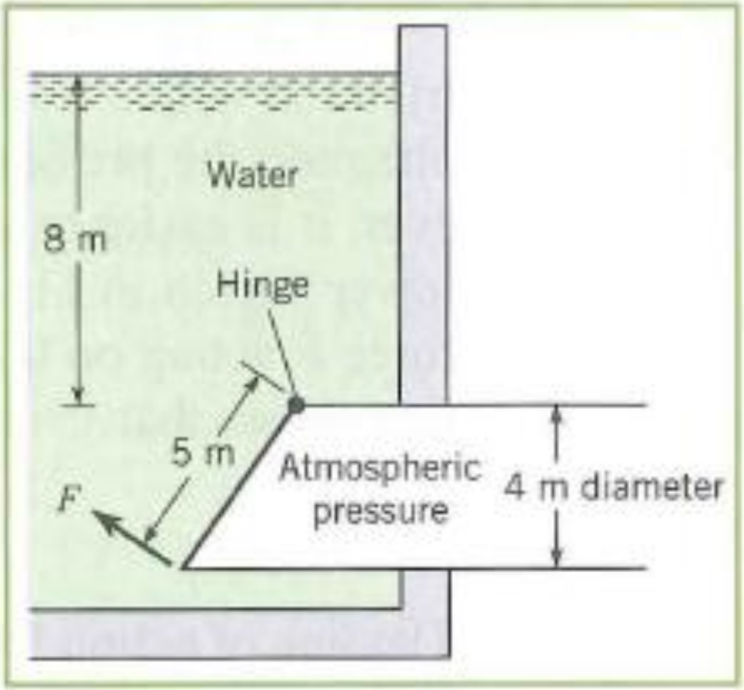
Situation: Water pressure is acting on an elliptical gate.

Find: Normal force (in newtons) required to open gate.

Properties: Water (10°C), Table A.5: $\gamma = 9810\ \text{N/m}^3$.

Assumptions:

1. Neglect the weight of the gate.
2. Neglect friction between the bottom on the gate and the pipe wall.



Solution

1. Hydrostatic (resultant) force

- $\bar{p}$ = pressure at depth of the centroid

$$\bar{p} = (\gamma_{\text{water}})(z_{\text{centroid}}) = (9810\ \text{N/m}^3)(10\ \text{m}) = 98.1\ \text{kPa}$$

- A = area of elliptical panel (using Fig. A.1 to find formula)

$$A = \pi ab$$
$$= \pi(2.5\ \text{m})(2\ \text{m}) = 15.71\text{m}^2$$

- Calculate resultant force

$$F_p = \bar{p}A = (98.1\ \text{kPa})(15.71\ \text{m}^2) = \boxed{1.54\ \text{MN}}$$

2. Center of pressure

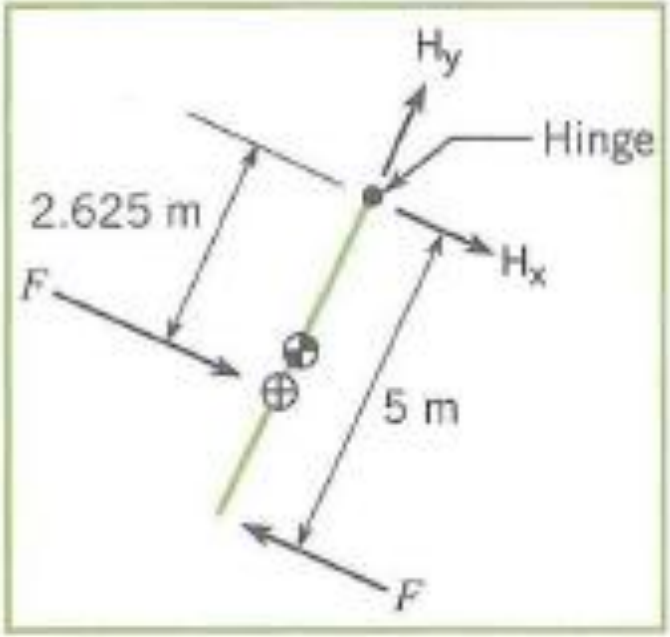
- $\bar{y} = 12.5$ m, where $\bar{y}$ is the slant distance from the water surface to the centroid.
- Area moment of inertia $\bar{I}$ of an elliptical panel using a formula from Fig. A.1

$$\bar{I} = \frac{\pi a^3 b}{4} = \frac{\pi (2.5 \text{ m})^3 (2 \text{ m})}{4} = 24.54 \text{ m}^4$$

- Finding center of pressure

$$y_{cp} - \bar{y} = \frac{\bar{I}}{\bar{y} A} = \frac{25.54 \text{ m}^4}{(12.5 \text{ m})(15.71 \text{m}^2)} = 0.125 \text{ m}$$

3. FBD of the gate:



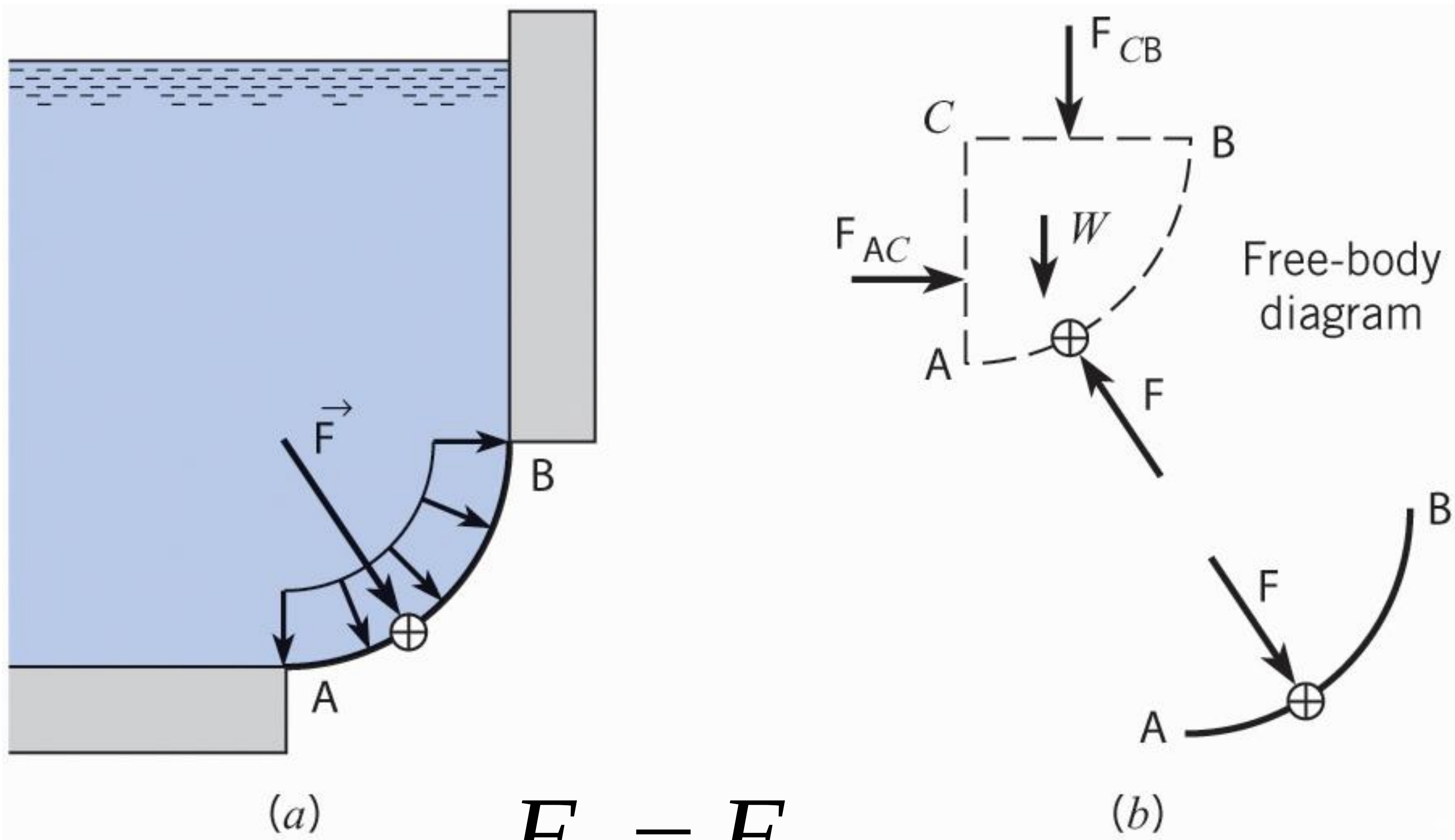
4. Moment equilibrium

$$\sum M_{\text{hinge}} = 0$$

$$1.541 \times 10^{6}\ \text{N} \times 2.625\ \text{m} - F \times 5\ \text{m} = 0$$

$$F = \boxed{809\ \text{kN}}$$

3.5 Forces on Curved Surfaces



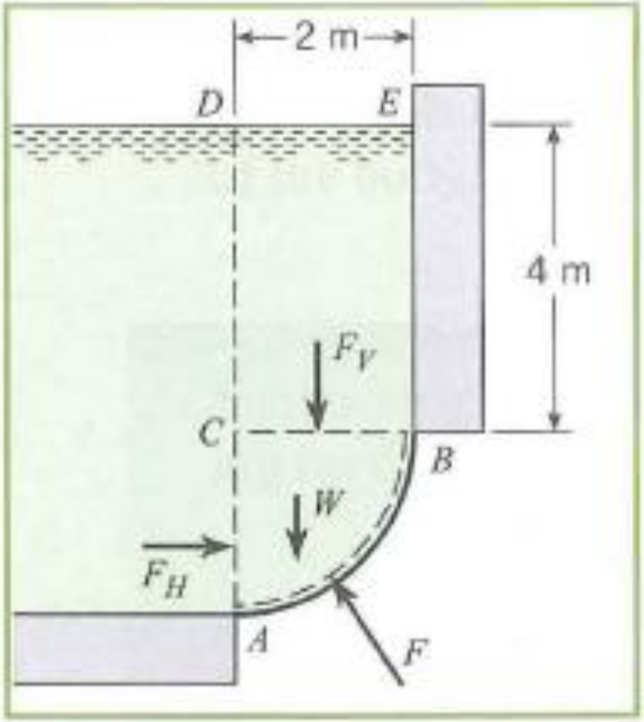
$$F_x = F_{AC}$$

$$F_y = F_{CB} + W$$

EXAMPLE 3.11 HYDROSTATIC FORCE ON A CURVED SURFACE

Sketch:

Surface *AB* is a circular arc with a radius of 2 m and a width of 1 m into the paper. The distance *EB* is 4 m. The fluid above surface *AB* is water, and atmospheric pressure prevails on the free surface of the water and on the bottom side of surface *AB*. Find the magnitude and line of action of the hydrostatic force acting on surface *AB*.



Solution

1. Equilibrium in the horizontal direction

$$F_x = F_H = \bar{p}A = (5\text{ m})(9810\text{ N/m}^3)(2\times 1\text{ m}^2)$$
$$= 98.1\text{ kN}$$

2. Equilibrium in the horizontal direction
   - Vertical force on side CB

$$F_V = \bar{p}_0 A = 9.81\text{ kN/m}^3 \times 4\text{ m}\times 2\text{ m}\times 1\text{ m} = 78.5\text{ kN}$$

   - Weight of the water in volume ABC

$$W = \gamma V_{ABC} = (\gamma)(\tfrac{1}{4}\pi r^2)(w)$$
$$= (9.81\text{ kN/m}^3)\times(0.25\times\pi\times 4\text{ m}^2)(1\text{ m}) = 30.8\text{ kN}$$

   - Summing forces

$$F_y = W + F_V = 109.3\text{ kN}$$

3. Line of action (horizontal force)

$$y_{cp} = \bar{y} + \frac{\bar{I}}{\bar{y}A} = (5\text{ m}) + \left(\frac{1\times 2^3/12}{5\times 2\times 1}\text{ m}\right)$$
$$y_{cp} = 5.067\text{ m}$$

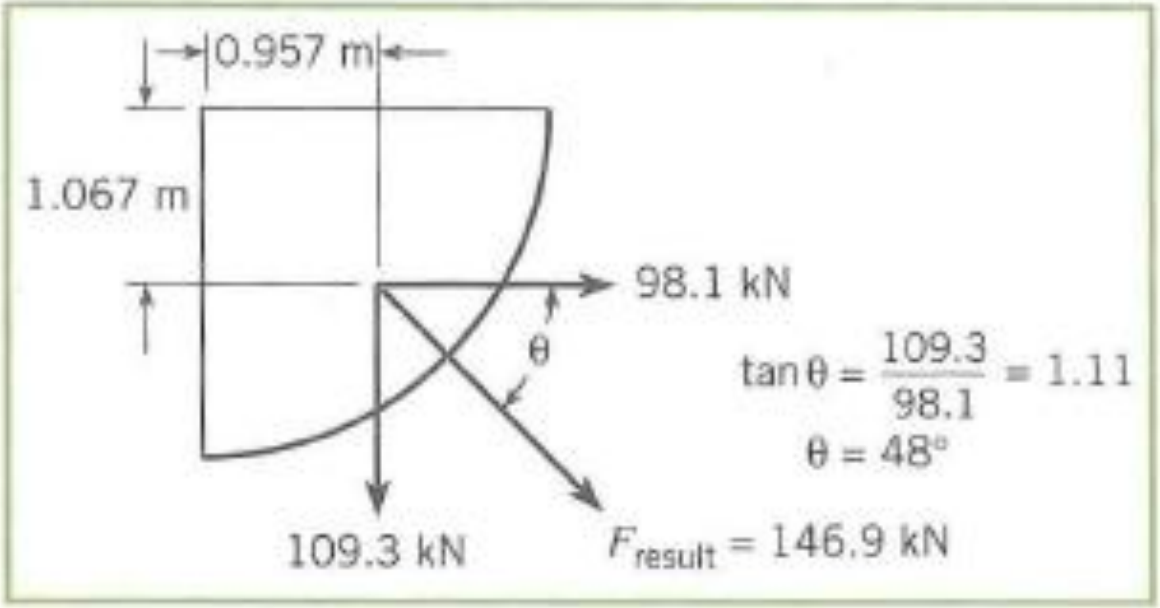
4. The line of action (x_{cp}) for the vertical force is found by summing moments about point C:

$$x_{cp}F_y = F_V \times 1\text{ m} + W \times \bar{x}_W$$

The horizontal distance from point C to the centroid of the area ABC is found using Fig. A.1: $\bar{x}_W = 4r/3\pi = 0.849$ m. Thus,

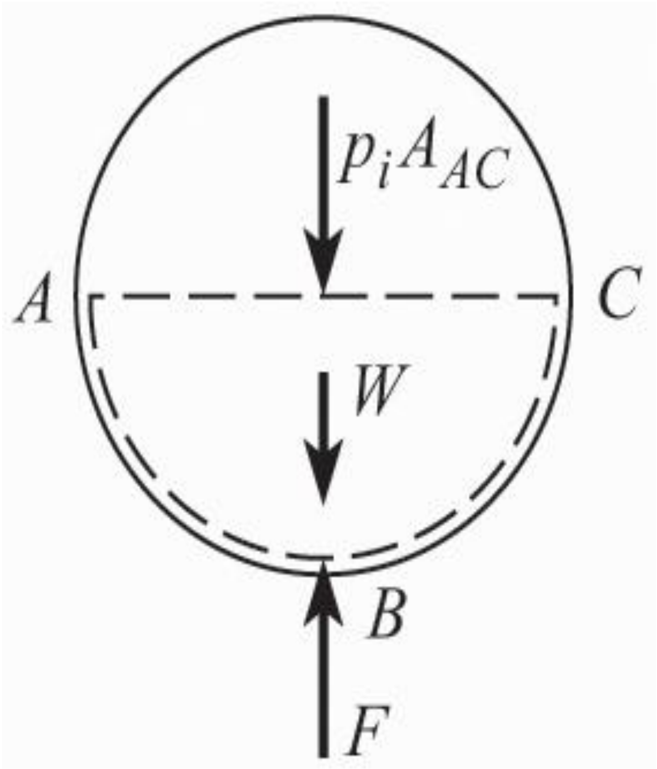
$$x_{cp} = \frac{78.5\text{ kN} \times 1\text{ m} + 30.8\text{ kN} \times 0.849\text{ m}}{109.3\text{ kN}} = 0.957\text{ m}$$

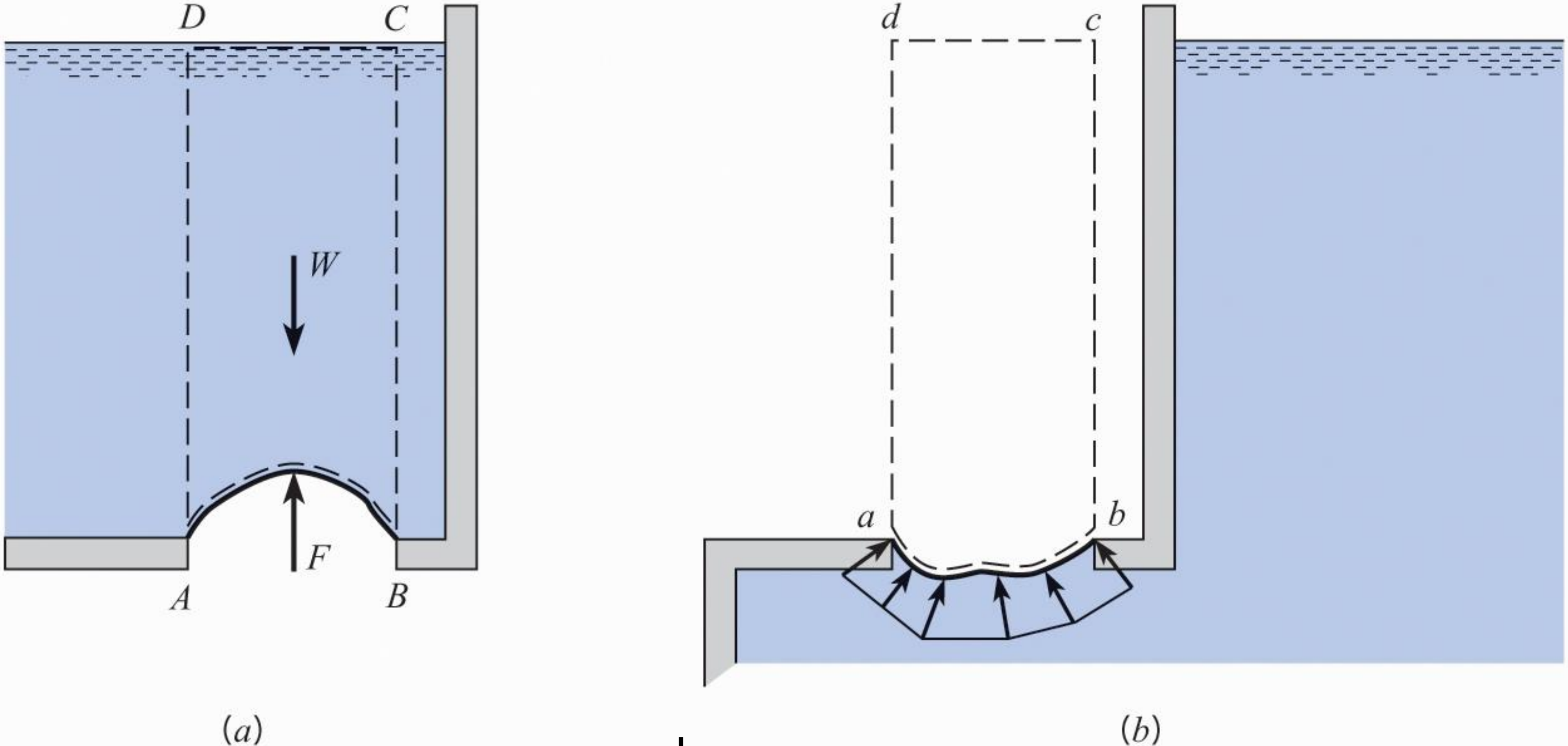
5. The resultant force that acts on the curved surface is shown in the following figure.



Hydrostatic force in a pressure vessel

$$F = p_i A_{AC} + W \approx p_i A_{AC}$$





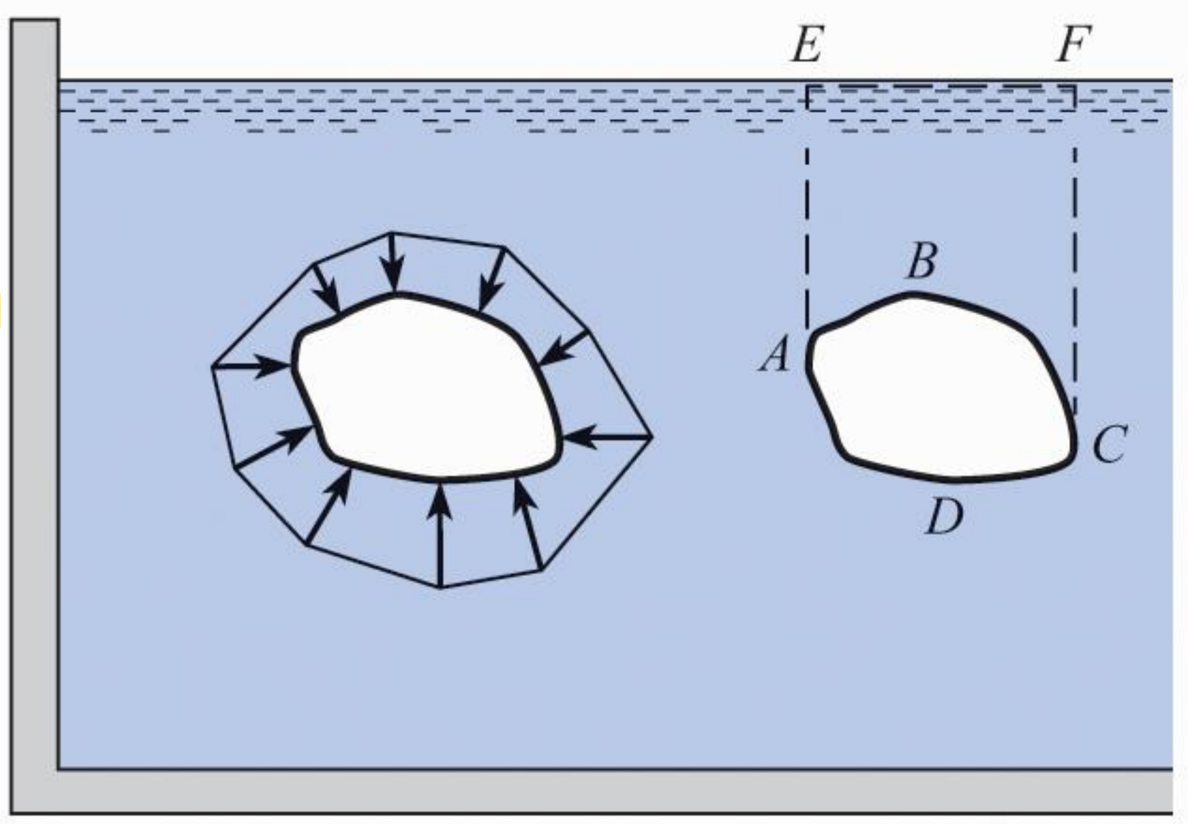
$$F = \gamma \forall_{ABCD} = W \downarrow$$

$$F = \gamma \forall_{abcd} = W \uparrow$$

If the curved surface is above the liquid, the weight of the liquid and vertical component of the hydrostatic force act in opposite directions.

3.6 Buoyancy

Archimedes Principle: The buoyant force acting on an immersed body in a fluid is equal to the weight of the fluid displaced by the body, and it acts upwards through the centroid of the displaced volume.



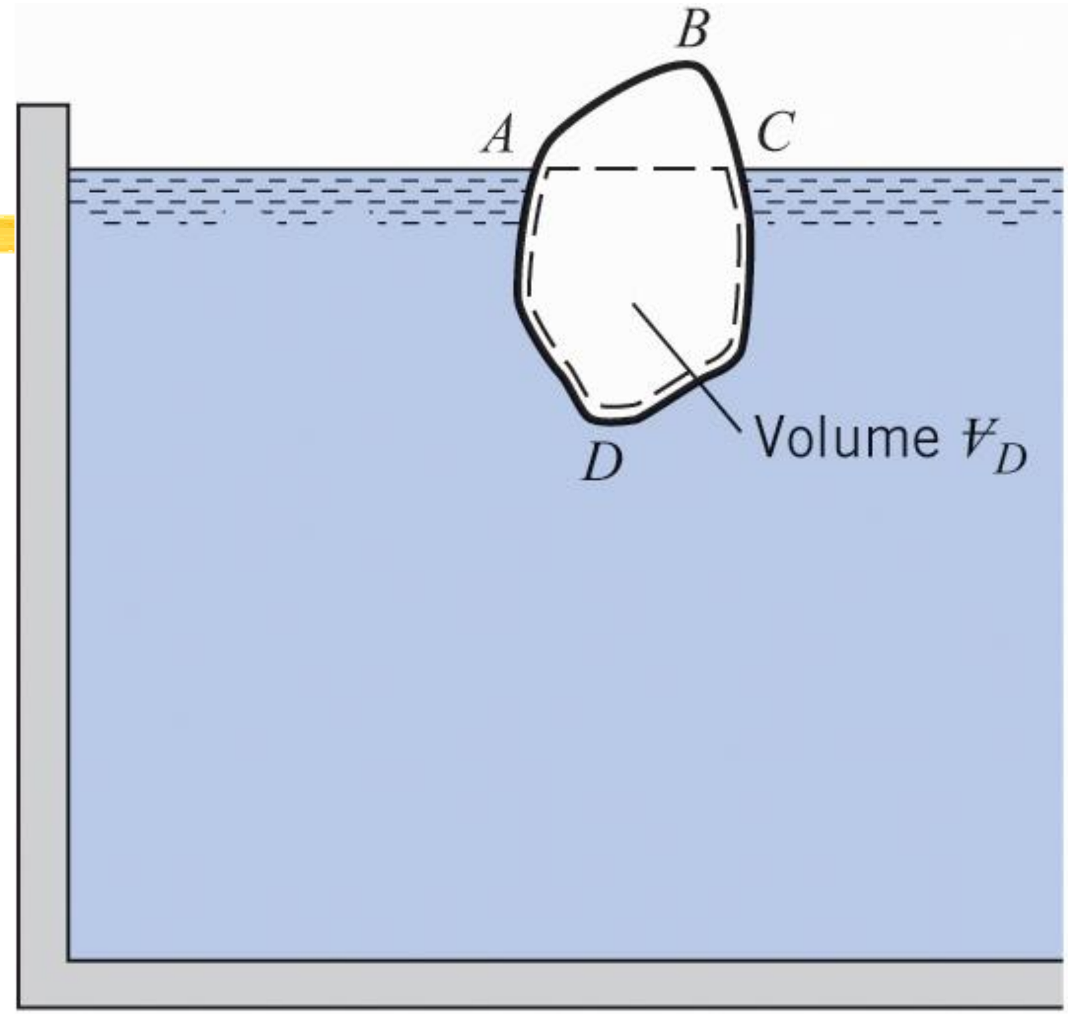
Submerged bodies

$$F_B = F_{up} - F_{down} = \gamma \forall_{EADCF} - \gamma \forall_{EABCF}$$

$$= \gamma \forall_b$$

Floating bodies

$$F_B = F_{up} = \gamma \forall_D$$



Hydrometer

A device to measure specific gravity of a fluid.

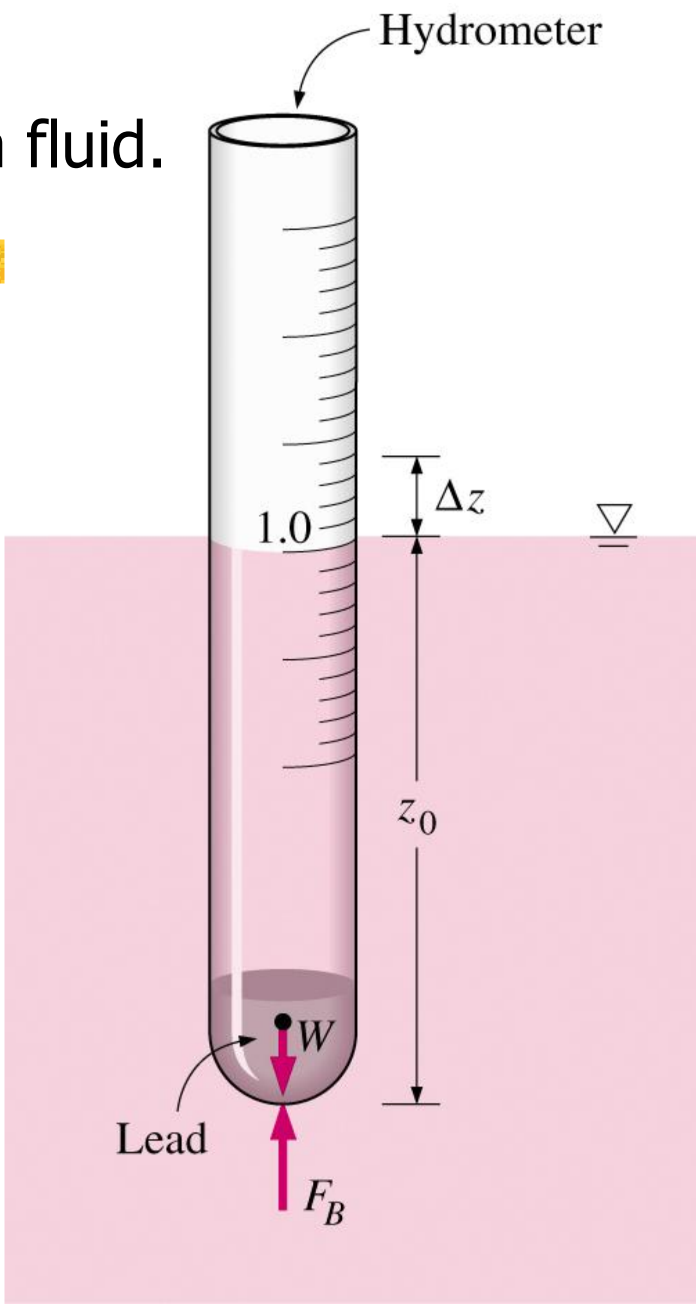
When hydrometer is immersed in pure water

$$W_{hydrometer} = F_{B,w} = \gamma_w A z_0$$

When hydrometer is immersed in another fluid

$$W_{hydrometer} = F_{B,f} = \gamma_f A(z_0 + \Delta z)$$

$$S_f = \frac{z_0}{z_0 + \Delta z}$$



EXAMPLE 3.12 BUOYANT FORCE ON A METAL PART

A metal part (object 2) is hanging by a thin cord from a floating wood block (object 1). The wood block has a specific gravity $S_1 = 0.3$ and dimensions of $50 \times 50 \times 10$ mm. The metal part has a volume of 6600 mm^3. Find the mass m_2 of the metal part and the tension T in the cord.

Problem Definition

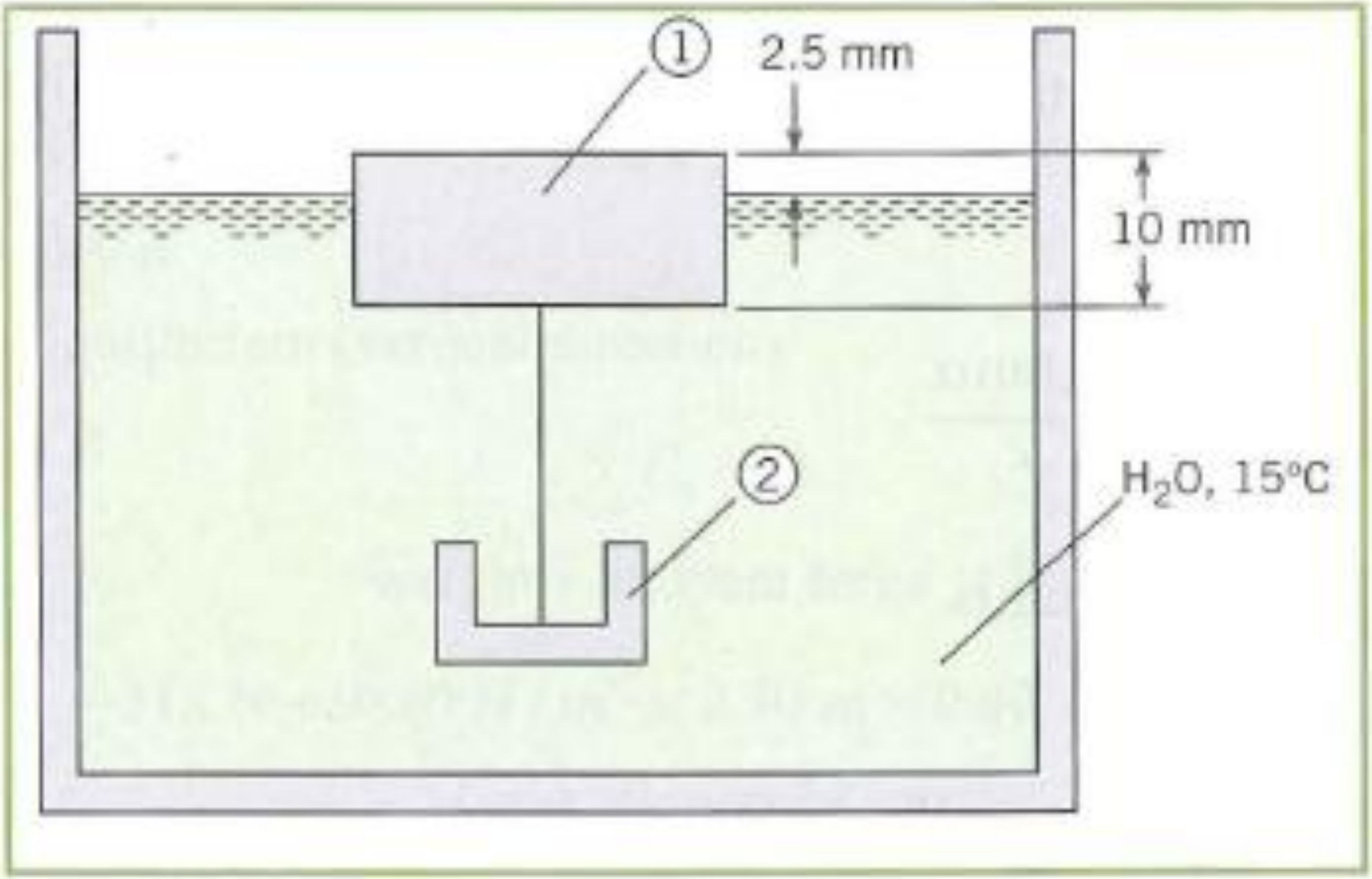
Situation: A metal part is suspended from a floating block of wood.

Find:

1. Mass (in grams) of the metal part.
2. Tension (in newtons) in the cord.

Properties:

1. Water (15°C), Table A.5: $\gamma = 9800$ N/m^3.
2. Wood: $S_1 = 0.3$.

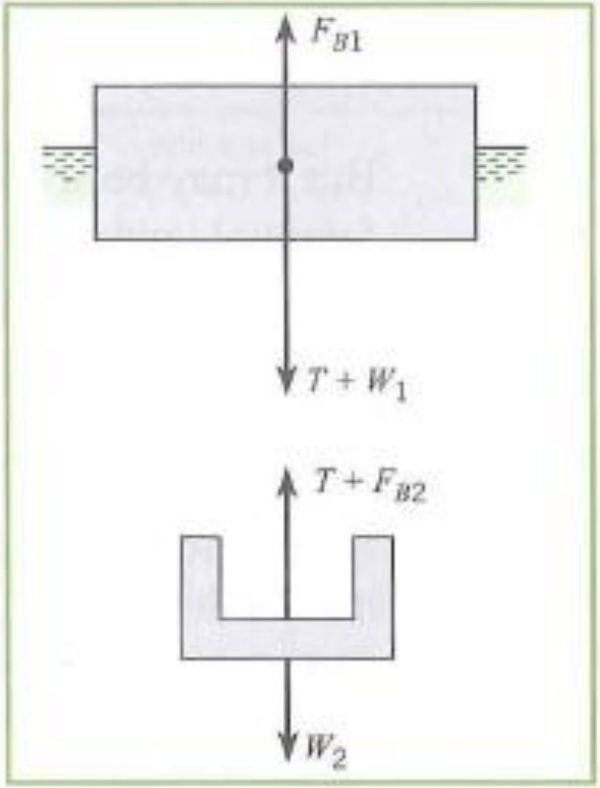


Plan

1. Draw FBDs of the block and the part.
2. Apply equilibrium to the block to find the tension.
3. Apply equilibrium to the part to find the weight of the part.
4. Calculate the mass of the metal part using $W = mg$.

Solution

1. FBDs



2. Force equilibrium (vertical direction) applied to block

$$T = F_{B1} - W_1$$

- Buoyant force $F_{B1} = \gamma V_{D1}$, where V_{D1} is the submerged volume

$$F_{B1} = \gamma V_{D1}$$
$$= (9800\ \mathrm{N/m^3})(50 \times 50 \times 7.5\ \mathrm{mm^3})(10^{-9}\ \mathrm{m^3/mm^3})$$
$$= 0.184\ \mathrm{N}$$

- Weight of the block

$$W_1 = \gamma S_1 \forall_1$$

$$= (9800\ \text{N/m}^3)(0.3)(50 \times 50 \times 10\ \text{mm}^3)(10^{-9}\ \text{m}^3/\text{mm}^3)$$

$$= 0.0735\ \text{N}$$

- Tension in the cord

$$T = (0.184 - 0.0735) = \boxed{0.110\ \text{N}}$$

3. Force equilibrium (vertical direction) applied to metal part
   - Buoyant force

$$F_{B2} = \gamma \forall_2 = (9800\ \text{N/m}^3)(6600\ \text{mm}^3)(10^{-9}) = 0.0647\ \text{N}$$

   - Equilibrium equation

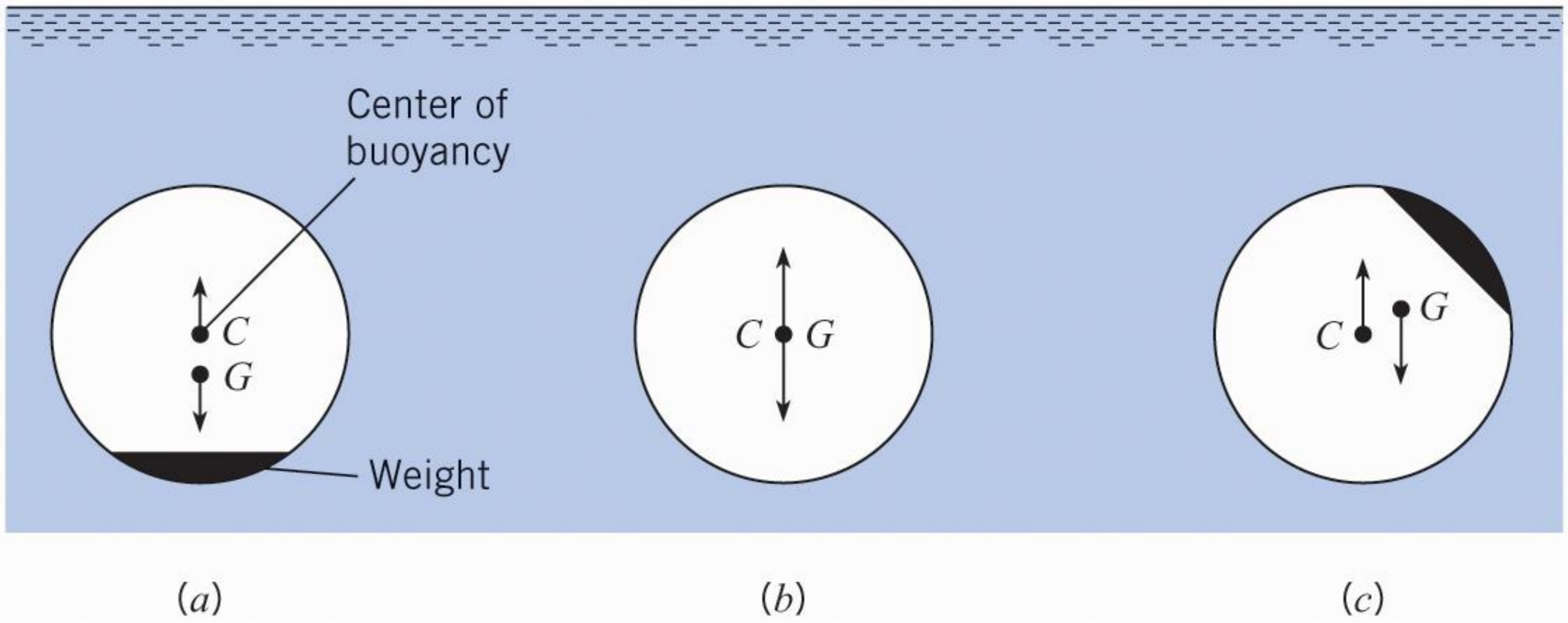
$$W_2 = T + F_{B2} = (0.110\ \text{N}) + (0.0647\ \text{N})$$

4. Mass of metal part

$$m_2 = W_2/g = \boxed{17.8\ \text{g}}$$

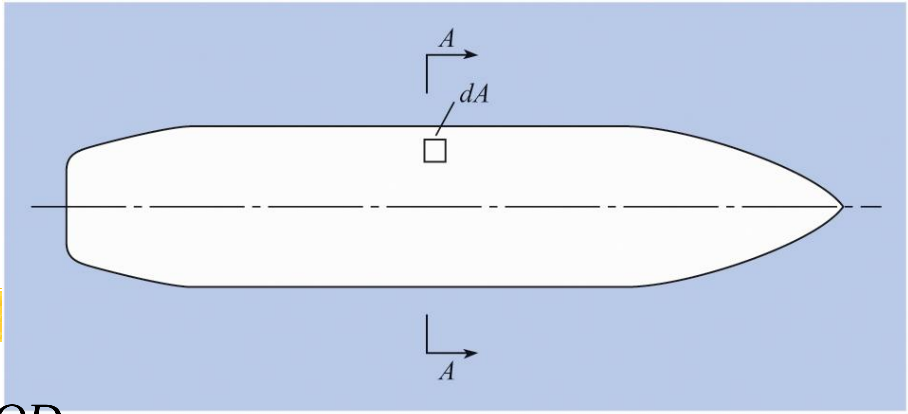
Stability of Immersed and Floating Bodies

Stability of Immersed Bodies

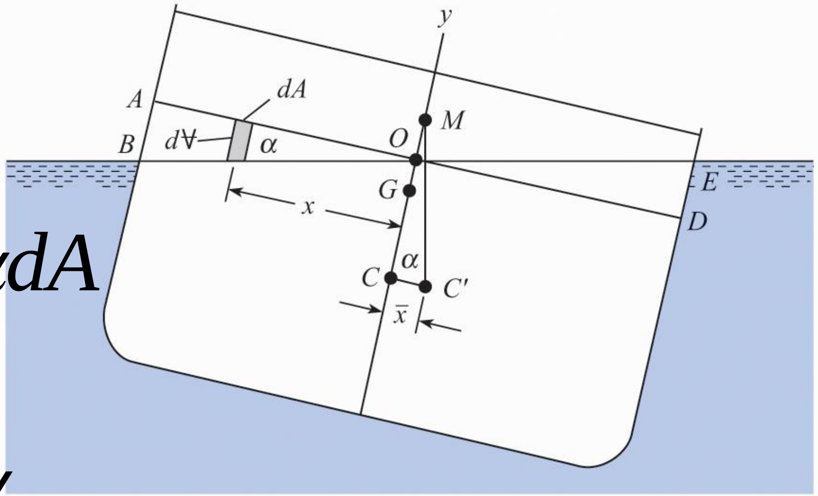


(a) Stable (b) Neutrally Stable (c) Unstable

Stability of floating Bodies



(a)



(b)

$$\bar{x}\forall = \sum x_i \Delta\forall_i \quad (\bar{x} = CC')$$

$$= \sum x_i \Delta\forall_{i,AOB} + \sum x_i \Delta\forall_{i,EOD}$$

$$= \int_{AOB} x d\forall + \int_{EOD} x d\forall$$

$$= \int_{AOB} x^2 \tan\alpha dA + \int_{EOD} x^2 \tan\alpha dA$$

$$= \tan\alpha \int_{A,waterline} x^2 dA = I_{00}\tan\alpha$$

$$\bar{x} = CC' = \frac{I_{00}\tan\alpha}{\forall} = CM\tan\alpha \Rightarrow CM = \frac{I_{00}}{\forall}$$

Metacentric height, *GM*

$$GM = CM - CG$$

$$= \frac{I_{00}}{\forall} - CG$$

If metacentric height is positive, the floating body is stable, if it is negative, the body is unstable.

EXAMPLE 3.13 STABILITY OF A FLOATING BLOCK

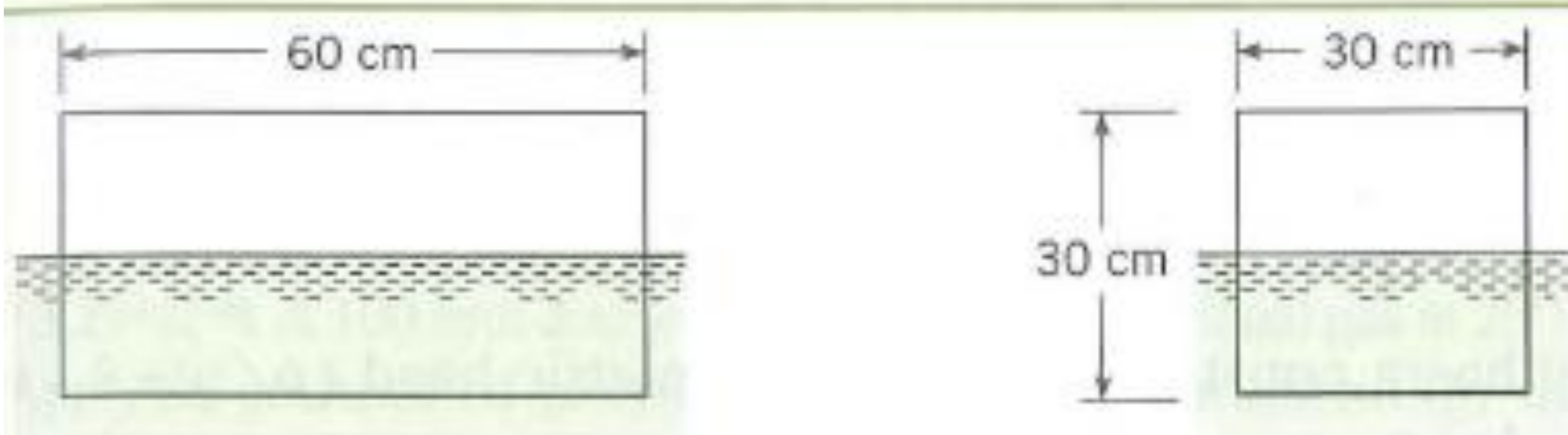
A block of wood 30 cm square in cross section and 60 cm long weighs 318 N. Will the block float with sides vertical as shown?

Problem Definition

Situation: A block of wood is floating in water.

Find: Is the block of wood stable?

Sketch:



Side view

End view

Plan

1. Apply force equilibrium to find the depth of submergence.
2. Determine if block is stable about the long axis by applying Eq. (3.42).
3. If block is not stable, repeat steps 1 and 2.

Solution

1. Equilibrium (vertical direction)

$$\sum F_y = 0$$

$$-\text{weight} + \text{buoyant force} = 0$$

$$-318 \text{ N} + 9810 \text{ N/m}^3 \times 0.30 \text{ m} \times 0.60 \text{ m} \times d = 0$$

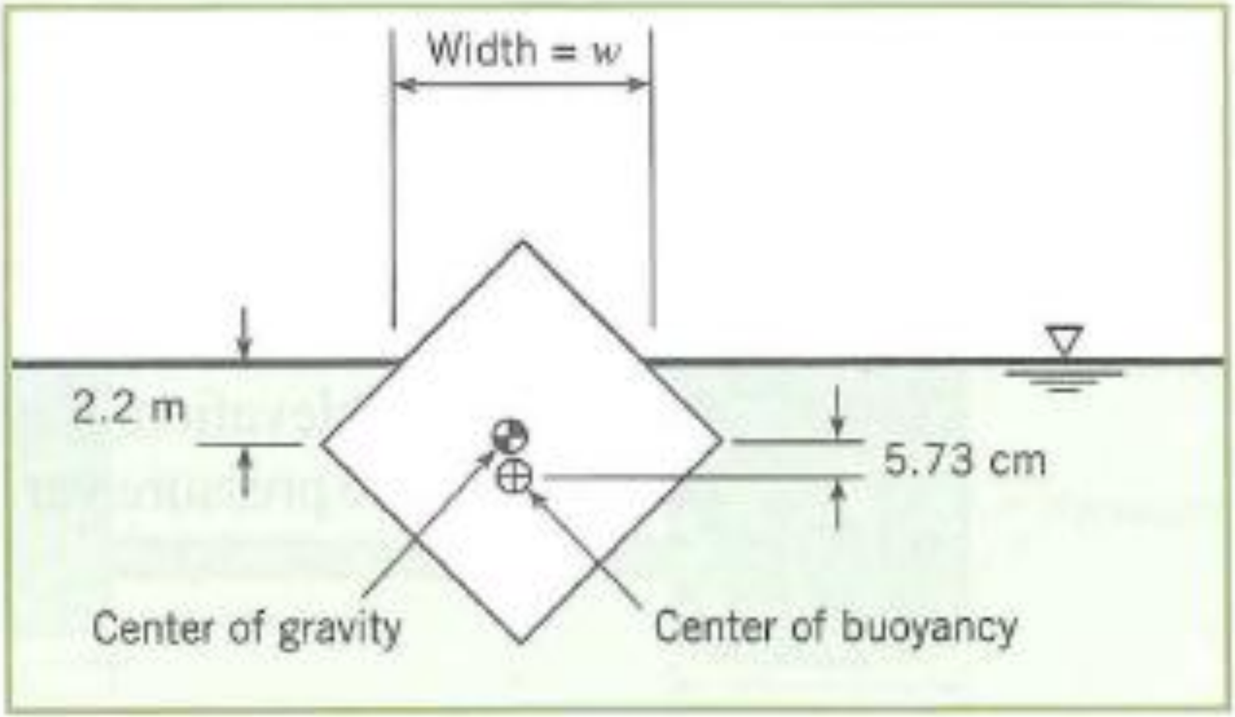
$$d = 0.18 \text{ m} = 18 \text{ cm}$$

2. Stability (longitudinal axis)

$$GM = \frac{I_{00}}{\forall} - CG = \frac{\frac{1}{12} \times 60 \times 30^3}{18 \times 60 \times 30} - (15 - 9)$$

$$= 4.167 - 6 = -1.833 \text{ cm}$$

Because the metacentric height is negative, the block is not stable about the longitudinal axis. Thus a slight disturbance will make it tip to the orientation shown below.



3. Equilibrium (vertical direction—see above figure)

$$-\text{weight} + \text{buoyant force} = 0$$

$$-(318\ \text{N}) + (9810\ \text{N/m}^3)(\forall_D) = 0$$

$$\forall_D = 0.0324\ \text{m}^3$$

4. Find the dimension w.

(Displaced volume) = (Block volume) – (Volume above the waterline).

$$\forall_D = 0.0324\ \text{m}^3 = (0.3^2)(0.6)\ \text{m}^3 - \frac{w^2}{4}(0.6\ \text{m})$$

$$w = 0.379\ \text{m}$$

5. Moment of inertia at the waterline

$$I_{00} = \frac{bh^3}{12} = \frac{(0.6\ \text{m})(0.379\ \text{m})^3}{12} = 0.00273\ \text{m}^4$$

6. Metacentric height

$$GM = \frac{I_{00}}{\forall} - CG = \frac{0.00273\ \text{m}^4}{0.0324\ \text{m}^3} - 0.0573\ \text{m} = 0.027\ \text{m}$$

Since the metacentric height is positive, the block will be stable in this position.