Chapter 10

Fourier Series

1

Periodic functions

A function $f(x)$ is called **periodic** if it is defined for all[2] real x and if there is some positive number p such that

$$f(x + p) = f(x) \qquad \text{for all } x. \tag{1}$$

This number p is called a **period** of $f(x)$. The graph of such a function is obtained by periodic repetition of its graph in any interval of length p (Fig. 236). Periodic phenomena and functions have many applications, as was mentioned before.

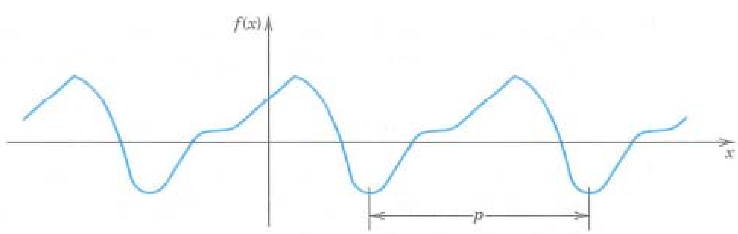

Fig. 236. Periodic function

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From (1) we have $f(x + 2p) = f[(x + p) + p] = f(x + p) = f(x)$, etc., and for any integer n,

(2) $$f(x + np) = f(x) \qquad \text{for all } x.$$

Hence $2p, 3p, 4p, \cdots$ are also periods of $f(x)$. Furthermore, if $f(x)$ and $g(x)$ have period p, then the function

$$h(x) = af(x) + bg(x) \qquad (a, b \text{ constant})$$

also has the period p.

If a periodic function $f(x)$ has a smallest period p (> 0), this is often called the **fundamental period** of $f(x)$. For $\cos x$ and $\sin x$ the fundamental period is 2π, for $\cos 2x$ and $\sin 2x$ it is π, and so on. A function without fundamental period is $f = const$.

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Trigonometric Series

Our problem in the first few sections of this chapter will be the representation of various functions of period $p = 2\pi$ in terms of the simple functions

(3) $$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \cdots, \quad \cos nx, \quad \sin nx, \cdots .$$

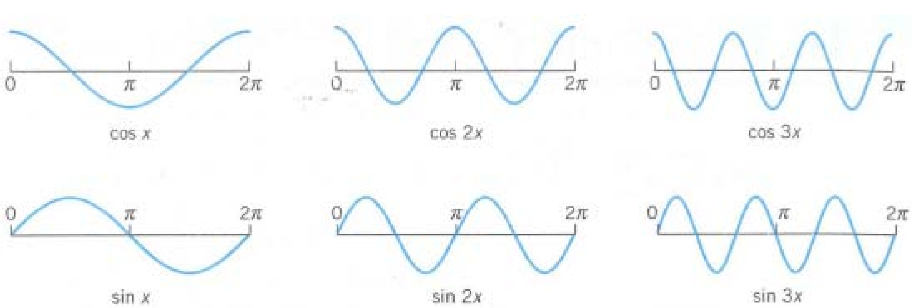

Fig. 237. Cosine and sine functions having the period 2π

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These functions have the period 2π. Figure 237 shows the first few of them.

The series that will arise in this connection will be of the form

$$(4) \qquad a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots,$$

where $a_0, a_1, a_2, \cdots, b_1, b_2, \cdots$ are real constants. Such a series is called a **trigonometric series,** and the a_n and b_n are called the **coefficients** of the series. Using the summation sign,[3] we may write this series

$$(4) \qquad \boxed{a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).}$$

The set of functions (3) from which we have made up the series (4) is often called the **trigonometric system,** to have a short name for it.

We see that each term of the series (4) has the period 2π. Hence *if the series (4) converges, its sum will be a function of period 2π.*

The point is that trigonometric series can be used for representing any practically important periodic function f, simple or complicated, of any period p. (This series will then be called the *Fourier series* of f.)

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Euler Formulas for the Fourier Coefficients

$$\text{(a)} \qquad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$\text{(b)} \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad n = 1, 2, \cdots,$$

$$\text{(c)} \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad n = 1, 2, \cdots.$$

These numbers given by (6) are called the **Fourier coefficients** of $f(x)$. The trigonometric series

$$(7) \qquad \boxed{a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

with coefficients given in (6) is called the **Fourier series** of $f(x)$ (regardless of convergence—we shall discuss this later in this section).

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Rectangular wave

Example 1

Find the Fourier coefficients of the periodic function $f(x)$ in Fig. 238a. The formula is

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$

Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc. (The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$.)

Solution. From (6a) we obtain $a_0 = 0$. This can also be seen without integration, since the area under the curve of $f(x)$ between $-\pi$ and π is zero. From (6b),

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\,\frac{\sin nx}{n}\Bigg|_{-\pi}^{0} + k\,\frac{\sin nx}{n}\Bigg|_{0}^{\pi}\right] = 0$$

because $\sin nx = 0$ at $-\pi$, 0, and π for all $n = 1, 2, \cdots$. Similarly, from (6c) we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[k\,\frac{\cos nx}{n}\Bigg|_{-\pi}^{0} - k\,\frac{\cos nx}{n}\Bigg|_{0}^{\pi}\right].$$

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Since $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, this yields

$$b_n = \frac{k}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right] = \frac{2k}{n\pi}(1 - \cos n\pi).$$

Now, $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, etc.; in general,

$$\cos n\pi = \begin{cases} -1 & \text{for odd } n, \\ 1 & \text{for even } n, \end{cases} \qquad \text{and thus} \qquad 1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n. \end{cases}$$

Hence the Fourier coefficients b_n of our function are

$$b_1 = \frac{4k}{\pi}, \qquad b_2 = 0, \qquad b_3 = \frac{4k}{3\pi}, \qquad b_4 = 0, \qquad b_5 = \frac{4k}{5\pi}, \cdots.$$

Since the a_n are zero, the Fourier series of $f(x)$ is

(8)
$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

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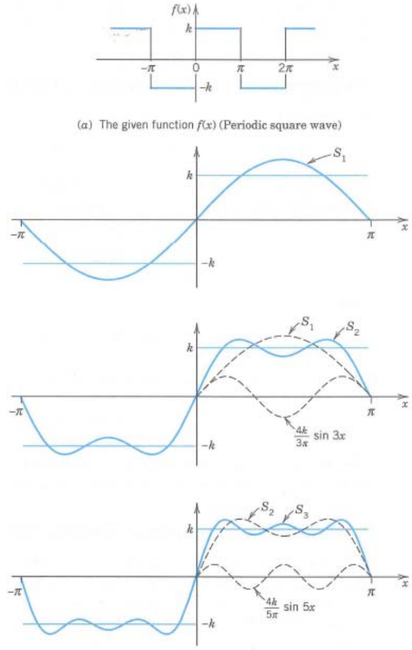

(*a*) The given function $f(x)$ (Periodic square wave)

(*b*) The first three partial sums of the corresponding Fourier series

Fig. 238. Example 1

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The partial sums are

$$S_1 = \frac{4k}{\pi} \sin x, \qquad S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right), \qquad \text{etc.,}$$

Their graphs in Fig. 238 seem to indicate that the series is convergent and has the sum $f(x)$, the given function. We notice that at $x = 0$ and $x = \pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the values $-k$ and k of our function.

Furthermore, assuming that $f(x)$ is the sum of the series and setting $x = \pi/2$, we have

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right),$$

thus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}.$$

This is a famous result by Leibniz (obtained in 1673 from geometrical considerations). It illustrates that the values of various series with constant terms can be obtained by evaluating Fourier series at specific points.

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Functions of Any Period $p = 2L$

The functions considered so far had period 2π, for simplicity. Of course, in applications, periodic functions will generally have other periods. But we show that the transition from period $p = 2\pi$ to period[9] $p = 2L$ is quite simple. It amounts to a stretch (or contraction) of scale on the axis.

If a function $f(x)$ of period $p = 2L$ has a **Fourier series,** we claim that this series is

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right) \tag{1}$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas**

(a) $$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

(b) $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \cdots$$

(c) $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \cdots.$$

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Periodic square wave

Find the Fourier series of the function (see Fig. 240)

$$f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < -1 \\ k & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < 2 \end{cases} \qquad p = 2L = 4, \quad L = 2.$$

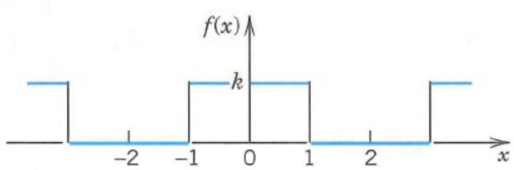

Fig. 240. Example 1

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Solution. From (2a) and (2b) we obtain

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\int_{-1}^{1} k\,dx = \frac{k}{2},$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}.$$

Thus $a_n = 0$ if n is even and

$$a_n = 2k/n\pi \quad \text{if} \quad n = 1, 5, 9, \cdots, \qquad a_n = -2k/n\pi \quad \text{if} \quad n = 3, 7, 11, \cdots.$$

From (2c) we find that $b_n = 0$ for $n = 1, 2, \cdots$. Hence the result is

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right).$$

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