Chapter 2

ENGR 6913 Advanced Engineering Mathematics

Textbook:

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2.1 Homogeneous Linear equations of 2nd Order

If $r(x) \equiv 0$ (that is, $r(x) = 0$ for all x considered; read "$r(x)$ is identically zero"), then (1) reduces to

(2) $$y'' + p(x)y' + q(x)y = 0$$

and is called **homogeneous.** If $r(x) \not\equiv 0$, then (1) is called **nonhomogeneous.** This is similar to Sec. 1.5.

For instance, a nonhomogeneous linear ODE is

$$y'' + 25y = e^{-x} \cos x,$$

and a homogeneous linear ODE is

$$xy'' + y' + xy = 0, \qquad \text{in standard form} \qquad y'' + \frac{1}{x} y' + y = 0.$$

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An example of a nonlinear ODE is

$$y''y + y'^2 = 0.$$

The functions p and q in (1) and (2) are called the **coefficients** of the ODEs. **Solutions** are defined similarly as for first-order ODEs in Chap. 1. A function

$$y = h(x)$$

is called a *solution* of a (linear or nonlinear) second-order ODE on some open interval I if h is defined and twice differentiable throughout that interval and is such that the ODE becomes an identity if we replace the unknown y by h, the derivative y' by h', and the second derivative y'' by h''. Examples are given below.

THEOREM 1

Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE (2), *any linear combination of two solutions on an open interval I is again a solution of* (2) *on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.*

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For a second-order homogeneous linear ODE (2) an **initial value problem** consists of (2) and two **initial conditions**

(4) $$y(x_0) = K_0, \qquad y'(x_0) = K_1.$$

These conditions prescribe given values K_0 and K_1 of the solution and its first derivative (the slope of its curve) at the same given $x = x_0$ in the open interval considered.

The conditions (4) are used to determine the two arbitrary constants c_1 and c_2 in a **general solution**

(5) $$y = c_1 y_1 + c_2 y_2$$

of the ODE; here, y_1 and y_2 are suitable solutions of the ODE.

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DEFINITION

General Solution, Basis, Particular Solution

A **general solution** of an ODE (2) on an open interval I is a solution (5) in which y_1 and y_2 are solutions of (2) on I that are not proportional, and c_1 and c_2 are arbitrary constants. These y_1, y_2 are called a **basis** (or a **fundamental system**) of solutions of (2) on I.

A **particular solution** of (2) on I is obtained if we assign specific values to c_1 and c_2 in (5).

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importance. Namely, two functions y_1 and y_2 are called **linearly independent** on an interval I where they are defined if

$$(7) \qquad k_1y_1(x) + k_2y_2(x) = 0 \quad \text{everywhere on } I \text{ implies} \quad k_1 = 0 \text{ and } k_2 = 0.$$

And y_1 and y_2 are called **linearly dependent** on I if (7) also holds for some constants k_1, k_2 not both zero. Then if $k_1 \neq 0$ or $k_2 \neq 0$, we can divide and see that y_1 and y_2 are proportional,

$$y_1 = -\frac{k_2}{k_1}y_2 \qquad \text{or} \qquad y_2 = -\frac{k_1}{k_2}y_1.$$

In contrast, in the case of linear *independence* these functions are not proportional because then we cannot divide in (7). This gives the following

DEFINITION

Basis (Reformulated)

A **basis** of solutions of (2) on an open interval I is a pair of linearly independent solutions of (2) on I.

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EXAMPLE 7

Reduction of Order if a Solution Is Known. Basis

Find a basis of solutions of the ODE

$$(x^2 - x)y'' - xy' + y = 0.$$

Solution. Inspection shows that $y_1 = x$ is a solution because $y_1' = 1$ and $y_1'' = 0$, so that the first term vanishes identically and the second and third terms cancel. The idea of the method is to substitute

$$y = uy_1 = ux, \qquad y' = u'x + u, \qquad y'' = u''x + 2u'$$

into the ODE. This gives

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0.$$

ux and $-xu$ cancel and we are left with the following ODE, which we divide by x, order, and simplify,

$$(x^2 - x)(u''x + 2u') - x^2u' = 0, \qquad (x^2 - x)u'' + (x - 2)u' = 0.$$

Continued

This ODE is of first order in $v = u'$, namely, $(x^2 - x)v' + (x - 2)v = 0$. Separation of variables and integration gives

$$\frac{dv}{v} = -\frac{x - 2}{x^2 - x}\,dx = \left(\frac{1}{x - 1} - \frac{2}{x}\right)dx, \qquad \ln|v| = \ln|x - 1| - 2\ln|x| = \ln\frac{|x - 1|}{x^2}.$$

We need no constant of integration because we want to obtain a particular solution; similarly in the next integration. Taking exponents and integrating again, we obtain

$$v = \frac{x - 1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \qquad u = \int v\,dx = \ln|x| + \frac{1}{x}, \qquad \text{hence} \qquad y_2 = ux = x\ln|x| + 1.$$

Since $y_1 = x$ and $y_2 = x\ln|x| + 1$ are linearly independent (their quotient is not constant), we have obtained a basis of solutions, valid for all positive x. ■

2.2 2nd Order Homogeneous equations with constant coefficients

We shall now consider second-order homogeneous linear ODEs whose coefficients a and b are constant,

(1) $$y'' + ay' + by = 0.$$

These equations have important applications, especially in connection with mechanical and electrical vibrations, as we shall see in Secs. 2.4, 2.8, and 2.9.

How to solve (1)? We remember from Sec. 1.5 that the solution of the *first-order* linear ODE with a constant coefficient k

$$y' + ky = 0$$

is an exponential function $y = ce^{-kx}$. This gives us the idea to try as a solution of (1) the function

Continued

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(2) $$y = e^{\lambda x}.$$

Substituting (2) and its derivatives

$$y' = \lambda e^{\lambda x} \qquad \text{and} \qquad y'' = \lambda^2 e^{\lambda x}$$

into our equation (1), we obtain

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

Hence if λ is a solution of the important **characteristic equation** (or *auxiliary equation*)

(3) $$\lambda^2 + a\lambda + b = 0$$

then the exponential function (2) is a solution of the ODE (1). Now from elementary algebra we recall that the roots of this quadratic equation (3) are

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(4) $$\lambda_1 = \tfrac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \qquad \lambda_2 = \tfrac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right).$$

(3) and (4) will be basic because our derivation shows that the functions

(5) $$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}$$

are solutions of (1). Verify this by substituting (5) into (1).

From algebra we further know that the quadratic equation (3) may have three kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, namely,

(Case I) *Two real roots if* $a^2 - 4b > 0$,
(Case II) *A real double root if* $a^2 - 4b = 0$,
(Case III) *Complex conjugate roots if* $a^2 - 4b < 0$.

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Case I. Two Distinct Real Roots λ_1 and λ_2

In this case, a basis of solutions of (1) on any interval is

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}$$

because y_1 and y_2 are defined (and real) for all x and their quotient is not constant. The corresponding general solution is

(6) $$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

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Case II. Real Double Root $\lambda = -a/2$

If the discriminant $a^2 - 4b$ is zero, we see directly from (4) that we get only one root, $\lambda = \lambda_1 = \lambda_2 = -a/2$, hence only one solution,

$$y_1 = e^{-(a/2)x}.$$

in the case of a double root of (3) a basis of solutions of (1) on any interval is

$$e^{-ax/2}, \qquad xe^{-ax/2}.$$

The corresponding general solution is

(7)
$$y = (c_1 + c_2 x)e^{-ax/2}.$$

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Case III. Complex Roots $-\frac{1}{2}a + i\omega$ and $-\frac{1}{2}a - i\omega$

This case occurs if the discriminant $a^2 - 4b$ of the characteristic equation (3) is negative. In this case, the roots of (3) and thus the solutions of the ODE (1) come at first out complex. However, we show that from them we can obtain a basis of *real* solutions

(8)
$$y_1 = e^{-ax/2} \cos \omega x, \qquad y_2 = e^{-ax/2} \sin \omega x \qquad (\omega > 0)$$

where $\omega^2 = b - \frac{1}{4}a^2$. It can be verified by substitution that these are solutions in the present case. We shall derive them systematically after the two examples by using the complex exponential function. They form a basis on any interval since their quotient $\cot \omega x$ is not constant. Hence a real general solution in Case III is

(9)
$$y = e^{-ax/2} (A \cos \omega x + B \sin \omega x) \qquad (A, B \text{ arbitrary}).$$

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Summary of Cases I–III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2}\cos\omega x$ $e^{-ax/2}\sin\omega x$	$y = e^{-ax/2}(A\cos\omega x + B\sin\omega x)$

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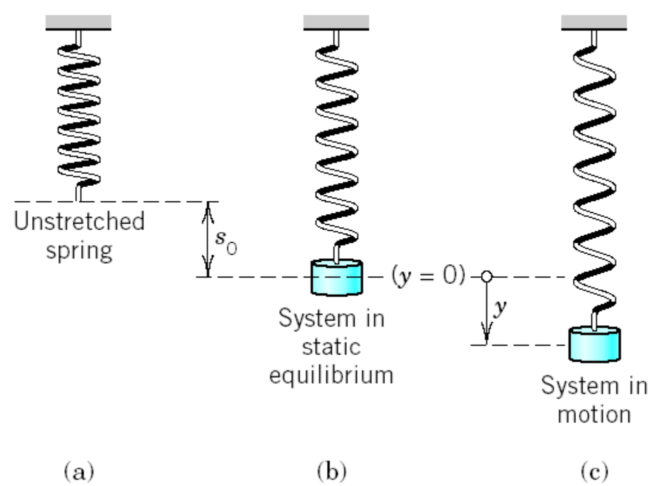
2.5 Modeling: Free Oscillations



Mechanical mass–spring system

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How can we obtain the motion of the body, say, the displacement $y(t)$ as function of time t? Now this motion is determined by **Newton's second law**

(1) $$\text{Mass} \times \text{Acceleration} = my'' = \text{Force}$$

where $y'' = d^2y/dt^2$ and "Force" is the resultant of all the forces acting on the body.

(For systems of units and conversion factors, see the inside of the front cover.)

We choose the ***downward direction as the positive direction,*** thus regarding downward forces as positive and upward forces as negative.

The spring is first unstretched. We now attach the body. This stretches the spring by an amount s_0 shown in the figure. It causes an upward force F_0 in the spring. Experiments show that F_0 is proportional to the stretch s_0, say,

(2) $$F_0 = -ks_0$$ **(Hooke's law[2]).**

k (> 0) is called the **spring constant** (or *spring modulus*). The minus sign indicates that F_0 points upward, in our negative direction. Stiff springs have large k. (Explain!)

The extension s_0 is such that F_0 in the spring balances the weight $W = mg$ of the body (where $g = 980 \text{ cm/sec}^2 = 32.17 \text{ ft/sec}^2$ is the gravitational constant). Hence $F_0 + W = -ks_0 + mg = 0$. These forces will not affect the motion. Spring and body are again at rest. This is called the **static equilibrium** of the system . We measure the displacement $y(t)$ of the body from this 'equilibrium point' as the origin $y = 0$, downward positive and upward negative.

From the position $y = 0$ we pull the body downward. This further stretches the spring by some amount $y > 0$ (the distance we pull it down). By Hooke's law this causes an (additional) upward force F_1 in the spring,

$$F_1 = -ky.$$

F_1 is a **restoring force.** It has the tendency to *restore* the system, that is, to pull the body back to $y = 0$.

Undamped System: ODE and Solution

Every system has damping—otherwise it would keep moving forever. But practically, the effect of damping may often be negligible, for example, for the motion of an iron ball on a spring during a few minutes. Then F_1 is the only force in (1) causing the motion. Hence (1) gives the model $my'' = -ky$ or

(3) $$my'' + ky = 0.$$

We obtain as a general solution

(4) $$y(t) = A \cos \omega_0 t + B \sin \omega_0 t, \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

The corresponding motion is called a **harmonic oscillation.**

Since the trigonometric functions in (4) have the period $2\pi/\omega_0$, the body executes $\omega_0/2\pi$ cycles per second. This is the **frequency** of the oscillation, which is also called the **natural frequency** of the system. It is measured in cycles per second. Another name for cycles/sec is hertz (Hz).[3]

The sum in (4) can be combined into a phase-shifted cosine with amplitude $C = \sqrt{A^2 + B^2}$ and phase angle $\delta = \arctan (B/A)$,

(4*) $$y(t) = C \cos (\omega_0 t - \delta).$$

Damped System: ODE and Solutions

We now add a **damping force**

$$F_2 = -cy'$$

to our model $my'' = -ky$, so that we have $my'' = -ky - cy'$ or

(5) $$my'' + cy' + ky = 0.$$

Physically this can be done by connecting the body to a dashpot; see Fig. 35. We assume this new force to be proportional to the velocity $y' = dy/dt$, as shown. This is generally a good approximation, at least for small velocities.

c is called the **damping constant.** We show that c is positive. If at some instant, y' is positive, the body is moving downward (which is the positive direction). Hence the damping force $F_2 = -cy'$, always acting *against* the direction of motion, must be an upward force, which means that it must be negative, $F_2 = -cy' < 0$, so that $-c < 0$ and $c > 0$. For an upward motion, $y' < 0$ and we have a downward $F_2 = -cy > 0$; hence $-c < 0$ and $c > 0$, as before.

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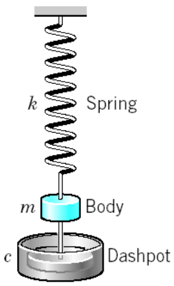
The ODE (5) is homogeneous linear and has constant coefficients. Hence we can solve it by the method in Sec. 2.2. The characteristic equation is (divide (5) by m)

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$



Damped system

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By the usual formula for the roots of a quadratic equation we obtain, as in Sec. 2.2,

$$\textbf{(6)} \quad \lambda_1 = -\alpha + \beta, \quad \lambda_2 = -\alpha - \beta, \quad \text{where} \quad \alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}.$$

It is now most interesting that depending on the amount of damping (much, medium, or little) there will be three types of motion corresponding to the three Cases I, II, II in Sec. 2.2:

Case I.	$c^2 > 4mk.$	*Distinct real roots* λ_1, λ_2.	**(Overdamping)**
Case II.	$c^2 = 4mk.$	*A real double root.*	**(Critical damping)**
Case III.	$c^2 < 4mk.$	*Complex conjugate roots.*	**(Underdamping)**

Case I. Overdamping

If the damping constant c is so large that $c^2 > 4mk$, then λ_1 and λ_2 are distinct real roots. In this case the corresponding general solution of (5) is

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}. \tag{7}$$

We see that in this case, damping takes out energy so quickly that the body does not oscillate. For $t > 0$ both exponents in (7) are negative because $\alpha > 0$, $\beta > 0$, and $\beta^2 = \alpha^2 - k/m < \alpha^2$. Hence both terms in (7) approach zero as $t \to \infty$. Practically speaking, after a sufficiently long time the mass will be at rest at the static equilibrium position ($y = 0$). Figure 36 shows (7) for some typical initial conditions.

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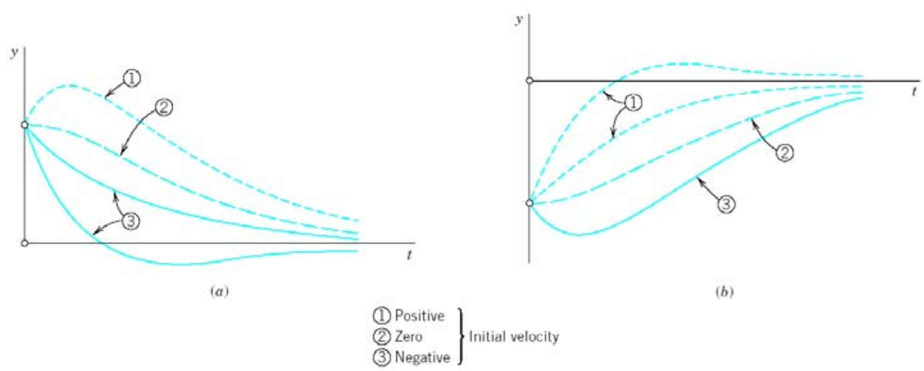
Case II. Critical Damping

Critical damping is the border case between nonoscillatory motions (Case I) and oscillations (Case III). It occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$,



Typical motions (7) in the overdamped case
(*a*) Positive initial displacement
(*b*) Negative initial displacement

Continued

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so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$. Then the corresponding general solution of (5) is

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}. \tag{8}$$

This solution can pass through the equilibrium position $y = 0$ at most once because $e^{-\alpha t}$ is never zero and $c_1 + c_2 t$ can have at most one positive zero. If both c_1 and c_2 are positive (or both negative), it has no positive zero, so that y does not pass through 0 at all. Figure 43 shows typical forms of (8). Note that they look almost like those in the previous figure.

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Case III. Underdamping

This is the most interesting case. It occurs if the damping constant c is so small that $c^2 < 4mk$. Then β in (6) is no longer real but pure imaginary, say,

$$\beta = i\omega^* \qquad \text{where} \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad (> 0). \tag{9}$$

$$\lambda_1 = -\alpha + i\omega^*, \qquad \lambda_2 = -\alpha - i\omega^*$$

with $\alpha = c/(2m)$, as given in (6). Hence the corresponding general solution is

$$y(t) = e^{-\alpha t}(A \cos \omega^* t + B \sin \omega^* t) = Ce^{-\alpha t} \cos (\omega^* t - \delta) \tag{10}$$

where $C^2 = A^2 + B^2$ and $\tan \delta = B/A$, as in (4*).

Continued

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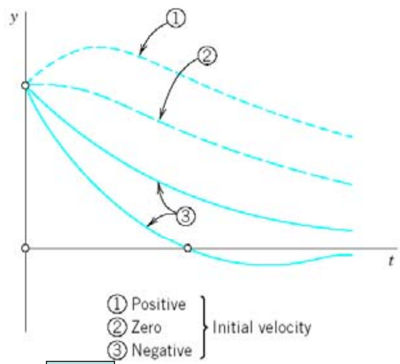
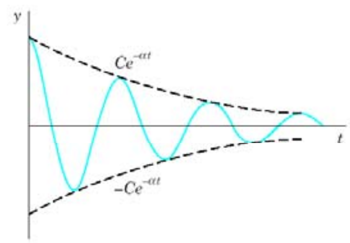
This represents **damped oscillations.** Their curve lies between the dashed curves $y = Ce^{-\alpha t}$ and $y = -Ce^{-\alpha t}$ in Fig. 38, touching them when $\omega^* t - \delta$ is an integer multiple of π because these are the points at which $\cos(\omega^* t - \delta)$ equals 1 or -1.

The frequency is $\omega^*/(2\pi)$ Hz (hertz, cycles/sec). From (9) we see that the smaller c (> 0) is, the larger is ω^* and the more rapid the oscillations become. If c approaches 0, then ω^* approaches $\omega_0 = \sqrt{k/m}$, giving the harmonic oscillation (4), whose frequency $\omega_0/(2\pi)$ is the natural frequency of the system.



Critical damping [see (8)]



Damped oscillation in Case III [see (10)]

2.8 Nonhomogeneous Equations

Method of Undetermined Coefficients

In this section we proceed from homogeneous to nonhomogeneous linear ODEs

(1) $$y'' + p(x)y' + q(x)y = r(x)$$

where $r(x) \not\equiv 0$. We shall see that a "general solution" of (1) is the sum of a general solution of the corresponding homogeneous ODE

(2) $$y'' + p(x)y' + q(x)y = 0$$

and a "particular solution" of (1). These two new terms "general solution of (1)" and "particular solution of (1)" are defined as follows.

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DEFINITION

General Solution, Particular Solution

A **general solution** of the nonhomogeneous ODE (1) on an open interval I is a solution of the form

(3) $$y(x) = y_h(x) + y_p(x);$$

here, $y_h = c_1y_1 + c_2y_2$ is a general solution of the homogeneous ODE (2) on I and y_p is any solution of (1) on I containing no arbitrary constants.

A **particular solution** of (1) on I is a solution obtained from (3) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

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THEOREM 1

Relations of Solutions of (1) to Those of (2)

(a) *The sum of a solution y of* (1) *on some open interval I and a solution $\tilde{y}$ of* (2) *on I is a solution of* (1) *on I. In particular,* (3) *is a solution of* (1) *on I.*

(b) *The difference of two solutions of* (1) *on I is a solution of* (2) *on I.*

THEOREM 2

A General Solution of a Nonhomogeneous ODE Includes All Solutions

If the coefficients $p(x)$, $q(x)$, and the function $r(x)$ in (1) *are continuous on some open interval I, then every solution of* (1) *on I is obtained by assigning suitable values to the arbitrary constants c_1 and c_2 in a general solution* (3) *of* (1) *on I.*

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2.9 Solution by Undetermined Coefficients

More precisely, the method of undetermined coefficients is suitable for linear ODEs with ***constant coefficients a and b***

(1) $$y'' + ay' + by = r(x)$$

when $r(x)$ is an exponential function, a power of x, a cosine or sine, or sums or products of such functions. These functions have derivatives similar to $r(x)$ itself. This gives the idea. We choose a form for y_p similar to $r(x)$, but with unknown coefficients to be determined by substituting that y_p and its derivatives into the ODE. Table 2.1 on p. 80 shows the choice of y_p for practically important forms of $r(x)$. Corresponding rules are as follows.

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Choice Rules for the Method of Undetermined Coefficients

(a) Basic Rule. *If $r(x)$ in (4) is one of the functions in the first column in Table 2.1, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into (4).*

(b) Modification Rule. *If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to (4), multiply your choice of y_p by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).*

(c) Sum Rule. *If $r(x)$ is a sum of functions in the first column of Table 2.1, choose for y_p the sum of the functions in the corresponding lines of the second column.*

The Basic Rule applies when $r(x)$ is a single term. The Modification Rule helps in the indicated case, and to recognize such a case, we have to solve the homogeneous ODE first. The Sum Rule follows by noting that the sum of two solutions of (1) with $r = r_1$ and $r = r_2$ (and the same left side!) is a solution of (1) with $r = r_1 + r_2$. (Verify!)

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Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

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2.10 Solution by Variation of Parameters

We continue our discussion of nonhomogeneous linear ODEs

(1) $$y'' + p(x)y' + q(x)y = r(x).$$

Lagrange's method gives a particular solution y_p of (1) on I in the form

(2) $$y_p(x) = -y_1 \int \frac{y_2 r}{W}\, dx + y_2 \int \frac{y_1 r}{W}\, dx$$

where y_1, y_2 form a basis of solutions of the corresponding homogeneous ODE

(3) $$y'' + p(x)y' + q(x)y = 0$$

on I, and W is the Wronskian of y_1, y_2,

(4) $$W = y_1 y_2' - y_2 y_1'$$

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EXAMPLE 1

Method of Variation of Parameters

Solve the nonhomogeneous ODE

$$y'' + y = \sec x = \frac{1}{\cos x}.$$

Solution. A basis of solutions of the homogeneous ODE on any interval is $y_1 = \cos x$, $y_2 = \sin x$. This gives the Wronskian

$$W(y_1, y_2) = \cos x \cos x - \sin x\,(-\sin x) = 1.$$

From (2), choosing zero constants of integration, we get the particular solution of the given ODE

$$\begin{aligned} y_p &= -\cos x \int \sin x \sec x \, dx + \sin x \int \cos x \sec x \, dx \\ &= \cos x \ln|\cos x| + x \sin x \end{aligned}.$$

Figure 69 shows y_p and its first term, which is small, so that $x \sin x$ essentially determines the shape of the curve of y_p. (Recall from Sec. 2.8 that we have seen $x \sin x$ in connection with resonance, except for notation.) From y_p and the general solution $y_h = c_1 y_1 + c_2 y_2$ of the homogeneous ODE we obtain the *answer*

$$y = y_h + y_p = (c_1 + \ln|\cos x|) \cos x + (c_2 + x) \sin x.$$

Continued

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