Chapter 1

ENGR 6913 Advanced Engineering Mathematics

Textbook:

Advanced Engineering Mathematics 8th Edition by Erwin Kreyszig

Ahmad Sana
sana@squ.edu.om

1

1.1 Basic Concepts and Ideas

In this chapter we shall consider **first-order ODEs.** Such equations contain only the first derivative y' and may contain y and any given functions of x. Hence we can write them as

(4) $$F(x, y, y') = 0$$

or often in the form

$$y' = f(x, y).$$

This is called the *explicit form,* in contrast with the *implicit form* (4). For instance, the implicit ODE $x^{-3}y' - 4y^2 = 0$ (where $x \neq 0$) can be written explicitly as $y' = 4x^3y^2$.

2

A function

$$y = h(x)$$

is called a **solution** of a given ODE (4) on some open interval $a < x < b$ if $h(x)$ is defined and differentiable throughout the interval and is such that the equation becomes an identity if y and y' are replaced with h and h', respectively. The curve (the graph) of h is called a **solution curve**.

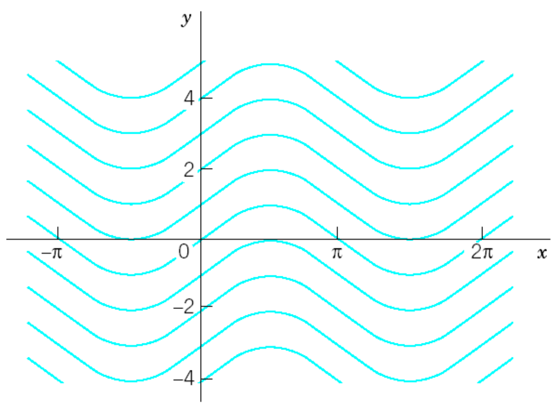
Here, **open interval** $a < x < b$ means that the endpoints a and b are not regarded as points belonging to the interval. Also, $a < x < b$ includes *infinite intervals* $-\infty < x < b$, $a < x < \infty$, $-\infty < x < \infty$ (the real line) as special cases.

3

Example 3

Solution Curves

The ODE $y' = dy/dx = \cos x$ can be solved directly by integration on both sides. Indeed, using calculus, we obtain $y = \int \cos x \, dx = \sin x + c$, where c is an arbitrary constant. This is a ***family of solutions.*** Each value of c, for instance, 2.75 or 0 or -8, gives one of these curves. Figure 2 shows some of them, for $c = -3, -2, -1, 0, 1, 2, 3, 4$.



4

Fig. 2. Solutions $y = \sin x + c$ of the ODE $y' = \cos x$

Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y(x_0) = y_0$, with given values x_0 and y_0, that is used to determine a value of the arbitrary constant c. Geometrically this condition means that the solution curve should pass through the point (x_0, y_0) in the xy-plane. An ODE together with an initial condition is called an **initial value problem.** Thus, if the ODE is explicit, $y' = f(x, y)$, the initial value problem is of the form

(10) $$y' = f(x, y), \qquad y(x_0) = y_0.$$

5

1.3 Separable Differential Equations

Many practically useful ODEs can be reduced to the form

(1) $$g(y)y' = f(x)$$

by purely algebraic manipulations. Then we can integrate on both sides with respect to x, obtaining

(2) $$\int g(y)\, y'\, dx = \int f(x)\, dx + c.$$

On the left we can switch to y as the variable of integration. By calculus, $y'\, dx = dy$, so that

(3) $$\int g(y)\, dy = \int f(x)\, dx + c.$$

Certain nonseparable ODEs can be made separable by transformations that introduce for y a new unknown function. We discuss this technique for a class of ODEs of practical importance, namely, for equations

$$y' = f\left(\frac{y}{x}\right). \tag{8}$$

Here, f is any (differentiable) function of y/x, such as $\sin(y/x)$, $(y/x)^4$, and so on. (Such an ODE is sometimes called a *homogeneous ODE,* a term we shall not use but reserve for a more important purpose in Sec. 1.5.)

The form of such an ODE suggests that we set $y/x = u$; thus,

(9) $\quad y = ux \quad$ and by product differentiation $\quad y' = u'x + u.$

Substitution into $y' = f(y/x)$ then gives $u'x + u = f(u)$ or $u'x = f(u) - u$. We see that this can be separated:

$$\frac{du}{f(u) - u} = \frac{dx}{x}. \tag{10}$$

7

1.4 Modeling: Separable Equations

Example 2

Mixing Problem

Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank in Fig. 9 contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissoved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time t.

Solution. ***Step 1. Setting up a model.*** Let $y(t)$ denote the amount of salt in the tank at time t. Its time rate of change is

$$y' = \text{Salt inflow rate} - \text{Salt outflow rate} \qquad \textbf{“Balance law”.}$$

5 lb times 10 gal gives an inflow of 50 lb of salt. Now, the outflow is 10 gal of brine. This is 10/1000 = 0.01 (= 1%) of the total brine content in the tank, hence 0.01 of the salt content $y(t)$, that is, $0.01y(t)$. Thus the model is the ODE

$$y' = 50 - 0.01y = -0.01(y - 5000). \tag{4}$$

8

Step 2. Solution of the model. The ODE (4) is separable. Separation, integration, and taking exponents on both sides gives

$$\frac{dy}{y - 5000} = -0.01\,dt, \qquad \ln|y - 5000| = -0.01t + c^*, \qquad y - 5000 = ce^{-0.01t}.$$

Initially the tank contains 100 lb of salt. Hence $y(0) = 100$ is the initial condition that will give the unique solution. Substituting $y = 100$ and $t = 0$ in the last equation gives $100 - 5000 = ce^0 = c$. Hence $c = -4900$. Hence the amount of salt in the tank at time t is

$$y(t) = 5000 - 4900e^{-0.01t}. \tag{5}$$

This function shows an exponential approach to the limit 5000 lb; see Fig. 9. Can you explain physically that $y(t)$ should increase with time? That its limit is 5000 lb? Can you see the limit directly from the ODE?

The model discussed becomes more realistic in problems on pollutants in lakes (see Problem Set 1.5, Prob. 27) or drugs in organs. These types of problems are more difficult because the mixing may be imperfect and the flow rates (in and out) may be different and known only very roughly. ■

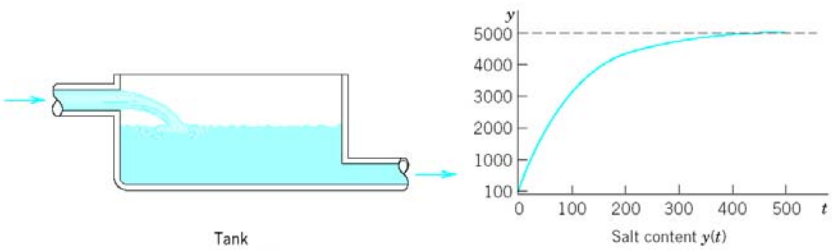


Fig. 9. Mixing problem in Example 3

9

1.5 Exact Differential Equations Integrating Factors

A first-order ODE $M(x, y) + N(x, y)y' = 0$, written as (use $dy = y'\,dx$ as in Sec. 1.3)

(1) $$M(x, y)\,dx + N(x, y)\,dy = 0$$

is called an **exact differential equation** if the **differential form** $M(x, y)\,dx + N(x, y)\,dy$ is **exact,** that is, this form is the differential

(2) $$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

of some function $u(x, y)$. Then (1) can be written

$$du = 0.$$

By integration we immediately obtain the general solution of (1) in the form

(3) $$u(x, y) = c.$$

(5) $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

This condition is not only necessary but also sufficient for (1) to be an exact differential equation. (We shall prove this in Sec. 10.2 in another context. Some calculus books (e.g., Ref. [GR11] also contain a proof.)

If (1) is exact, the function $u(x, y)$ can be found by inspection or in the following systematic way. From (4a) we have by integration with respect to x

(6) $$u = \int M\,dx + k(y);$$

in this integration, y is to be regarded as a constant, and $k(y)$ plays the role of a "constant" of integration. To determine $k(y)$, we derive $\partial u/\partial y$ from (6), use (4b) to get dk/dy, and integrate dk/dy to get k.

11

How to Find Integrating Factors

We multiply a given nonexact equation

(12) $$P(x, y)\,dx + Q(x, y)\,dy = 0,$$

by a function F that, in general, will be a function of both x and y. We want the result to be an exact equation

(13) $$FP\,dx + FQ\,dy = 0$$

so we can solve it as just discussed. Such a function $F(x, y)$ is then called an integrating factor of (12).

12

We let

$$(16) \qquad \frac{1}{F}\frac{dF}{dx} = R, \qquad \text{where} \qquad R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right).$$

THEOREM 1

Integrating Factor $F(x)$

If (12) is such that the right side R of (16), depends only on x, then (12) has an integrating factor $F = F(x)$, which is obtained by integrating (16) and taking exponents on both sides,

$$(17) \qquad F(x) = \exp \int R(x)\, dx.$$

THEOREM 2

Integrating Factor $F^*(y)$

If (12) is such that the right side R^* of (18) depends only on y, then (12) has an integrating factor $F^* = F^*(y)$, which is obtained from (18) in the form

$$(19) \qquad F^*(y) = \exp \int R^*(y)\, dy.$$

14

1.6 Linear Differential Equations
Bernoulli Equation

(1) $$y' + p(x)y = r(x).$$

(2) $$y' + p(x)y = 0$$ is called **homogeneous**.

The general solution of the homogeneous ODE (2),

(3) $$y(x) = ce^{-\int p(x)\,dx} \qquad (c = \pm e^{c^*} \quad \text{when} \quad y \gtrless 0);$$

here we may also choose $c = 0$ and obtain the **trivial solution** $y(x) = 0$ for all x in that interval.

Solution of nonhomogeneous linear ODE (1)

(4) $$y(x) = e^{-h}\left(\int e^h r\,dx + c\right), \qquad h = \int p(x)\,dx.$$

15

EXAMPLE 2

First-Order ODE, Initial Value Problem

Solve the initial value problem

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

Solution. Here $p = \tan x$, $r = \sin 2x = 2 \sin x \cos x$, and

$$\int p\,dx = \int \tan x\,dx = \ln|\sec x|.$$

From this we see that in (4),

$$e^h = \sec x, \qquad e^{-h} = \cos x, \qquad e^h r = (\sec x)(2 \sin x \cos x) = 2 \sin x,$$

and the general solution of our equation is

$$y(x) = \cos x \left(2\int \sin x\,dx + c\right) = c \cos x - 2\cos^2 x.$$

From this and the initial condition, $1 = c \cdot 1 - 2 \cdot 1^2$; thus $c = 3$ and the solution of our initial value problem is $y = 3 \cos x - 2 \cos^2 x$. Here $3 \cos x$ is the response to the initial data, and $-2 \cos^2 x$ is the response to the input $\sin 2x$. ■

16

Reduction to Linear Form. Bernoulli Equation

Numerous applications can be modeled by ODEs that are nonlinear but can be transformed to linear ODEs. One of the most useful ones of these is the **Bernoulli equation**[5]

$$y' + p(x)y = g(x)y^a \qquad (a \text{ any real number}). \tag{6}$$

If $a = 0$ or $a = 1$, Equation (6) is linear. Otherwise it is nonlinear. Then we set

$$u(x) = [y(x)]^{1-a}.$$

We differentiate this and substitute y' from (6), obtaining

$$u' = (1 - a)y^{-a}y' = (1 - a)y^{-a}(gy^a - py).$$

Simplification gives

$$u' = (1 - a)(g - py^{1-a}),$$

where $y^{1-a} = u$ on the right, so that we get the linear ODE

$$u' + (1 - a)pu = (1 - a)g. \tag{7}$$

Example 5

Logistic Equation

Solve the following Bernoulli equation, known as the **logistic equation** (or **Verhulst equation**[6]):

$$y' = Ay - By^2 \tag{8}$$

Solution. Write (8) in the form (6), that is,

$$y' - Ay = -By^2$$

to see that $a = 2$, so that $u = y^{1-a} = y^{-1}$. Differentiate this u and substitute y' from (8),

$$u' = -y^{-2}y' = -y^{-2}(Ay - By^2) = B - Ay^{-1}.$$

The last term is $-Ay^{-1} = -Au$. Hence we have obtained the linear ODE

$$u' + Au = B.$$

The general solution is [by (4)]

$$u = ce^{-At} + B/A.$$

Since $u = 1/y$, this gives the general solution of (8).

18

(9) $$y = \frac{1}{u} = \frac{1}{ce^{-At} + B/A}$$ (Fig. 18).

Directly from (8) we see that $y \equiv 0$ ($y(t) = 0$ for all t) is also a solution.

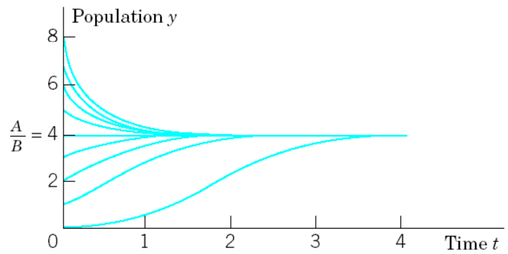


Fig. 18. Logistic population model. Curves (9) in Example 4 with $A/B = 4$

19

End of Chapter 1

20